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MEAN CURVATURE FLOW OF SURFACES IN EINSTEIN FOUR-MANIFOLDS

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Abstract

Let Σ be a compact oriented surface immersed in a four dimensional Kähler-Einstein manifold (M, ω). We consider the evolution of Σ in the direction of its mean curvature vector. It is proved that being symplectic is preserved along the flow and the flow does not develop type I singularity. When M has two parallel Kähler forms ω' and ω'' that determine different orientations and Σ is symplectic with respect to both ω' and ω'', we prove the mean curvature flow of Σ exists smoothly for all time. In the positive curvature case, the flow indeed converges at infinity.

1. Introduction

Let (M, g) be a Riemannian manifold and let α be a calibrating k-form on M i.e., $d\alpha = 0$ and $\text{comass}(\alpha) = 1$. In this article, we shall assume additionally α is parallel. This in particular implies M is of special holonomy.

A k-dimensional submanifold is said to be calibrated by α if the restriction of α gives the volume form of the submanifold. A simple application of Stokes' theorem shows a calibrated submanifold minimizes the volume functional in its homology class. To produce a calibrated submanifold, it is thus natural to consider the gradient flow of the volume functional. By the first variation formula of volume, this is equivalent to evolving a submanifold Σ_0 in the direction of its mean curvature vector. To make it precise, the mean curvature flow is the solution of the following system of parabolic equations.

$$\frac{dF}{dt}(x, t) = H(x, t)$$

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where $F : \Sigma \times [0, T) \to M$ is a one parameter family of immersions $F_t(\cdot) = F(\cdot, t)$ of Σ into M. $H(x, t)$ is the mean curvature vector of $F_t(\Sigma)$ at $F_t(x)$. We say F is the mean curvature flow of the immersed submanifold $F_0(\Sigma)$. For a fixed t, the submanifold $F_t(\Sigma)$ is denoted by Σ_t. If we assume $M = \mathbb{R}^n$, then in terms of coordinate $x^1, \cdots, x^k$ on Σ, the mean curvature flow is the following system of parabolic equations

$$F = F^A(x^1, \cdots, x^k, t),\ A = 1, \dots, n$$

$$\frac{\partial F^A}{\partial t} = \sum_{i,j,B} g^{ij}\, P^A_B\, \frac{\partial^2 F^B}{\partial x^i \partial x^j},\ A = 1, \dots, n$$

where g^{ij} is the inverse matrix to $g_{ij} = \frac{\partial F^A}{\partial x^i}\frac{\partial F^A}{\partial x^j}$ and $P^A_B = \delta^A_B - g^{kl}\frac{\partial F^A}{\partial x^k}\frac{\partial F^B}{\partial x^l}$ is the projection to the normal part.

The mean curvature flow of hypersurfaces has been studied extensively in the last decade. In this case, the mean curvature H is essentially a scalar function and the positivity of H is preserved along the flow. Very little is known in higher codimension except for the curve flows.

This article considers the next simplest higher codimension mean curvature flow, namely a surface flow in a four dimensional manifold. We impose a positivity condition on the initial submanifold. An oriented submanifold Σ is said to be almost calibrated by α if $*\alpha > 0$ where $*$ is the Hodge star operator on Σ.

The following question arises naturally. Can an almost calibrated submanifold be deformed to a calibrated one along the mean curvature flow? We study this question in the case when M is a four-dimensional Einstein manifold and Σ_0 is almost calibrated by a parallel calibrating form. When M is a Kähler-Einstein surface and the calibrating form is the Kähler form, an almost calibrated surface is a symplectic curve with the induced symplectic structure. A calibrated submanifold in this case is a holomorphic curve.

We use blow up analysis to characterize the singularities of mean curvature flow of symplectic surfaces. It turns out they are all so-called type II singularities.

Theorem A. *Let M be a four-dimensional Kähler-Einstein manifold, then a symplectic surface remains symplectic along the mean curvature flow and the flow does not develop any type* I *singularities.*

When M is locally a product and the initial surface is almost calibrated by two calibrating forms, we prove the following long time existence theorem.

Theorem B. *Let M be an oriented four-dimensional Einstein manifold with two parallel calibrating forms ω', ω'' such that ω' is self-dual and ω'' is anti-self-dual. If Σ is a compact oriented surface immersed in M such that $*\omega', *\omega'' > 0$ on Σ, then the mean curvature flow of Σ exists smoothly for all time.*

We remark that the assumption implies M is locally a product of two surfaces. As for convergence at infinity, we prove the following theorem in the nonnegative curvature case.

Theorem C. *Under the same assumption as in Theorem B. When M has nonnegative curvature, there exists a constant $1 > \epsilon > 0$ such that if Σ is a compact oriented surface immersed in M with $*\omega', *\omega'' > 1 - \epsilon$ on Σ, the mean curvature flow of Σ converges smoothly to a totally geodesic surface at infinity.*

This is proved by an uniform estimate of the norm of the second fundamental form.

When $M = S^2 \times S^2$, the combination of Theorem B and C yields:

Theorem D. *Let $M = (S^2, \omega_1) \times (S^2, \omega_2)$. If Σ is a compact oriented surface embedded in M such that $*\omega_1 > |*\omega_2|$. Then the mean curvature flow of Σ exists for all time and converges smoothly to an $S^2 \times \{p\}$.*

This theorem in particular applies to the graph of maps between two Riemann surfaces. Namely, let $f : (\Sigma_1, \omega_1) \to (\Sigma_2, \omega_2)$ be a map between Riemann surfaces of the same constant curvature and ω_i is the volume form of Σ_i. We consider the product $M = \Sigma_1 \times \Sigma_2$ and let $\omega' = \omega_1 + \omega_2$ and $\omega'' = \omega_1 - \omega_2$. If the Jacobian of f is less than one, then we have $*\omega' > 0$ and $*\omega'' > 0$ on the graph of f. Therefore this formulation gives a natural way to deform the map f to a constant map.

Corollary D. *Any smooth map between two-spheres with Jacobian less than one deforms to a constant map through the mean curvature flow of the graph.*

The article is organized as follows. In Section 2, the parabolic equation satisfied by a general parallel form along the mean curvature flow is derived. Section 3 discusses general calibrating two-forms in a four dimensional space. Section 4 computes the equation satisfied by a Kähler form along the mean curvature flow. Section 5 studies the singularities of mean curvature flow of symplectic surfaces and proves Theorem A. Section 6 concerns long time existence and Theorem B is proved there. Convergence at infinity is discussed in Section 7. Theorem C is proved at the end of this section. Section 8 discusses applications in the positive curvature case and proves Theorem D.

This project began in the fall of 1998 in an attempt to answer Professor S.-T. Yau's question, "How can a symplectic submanifold be deformed to a holomorphic one." Theorem A, in particular the result "symplectic remains symplectic" and the exclusion of type I singularity, was obtained in the summer of 1999. It has been presented in the geometry seminars at Stanford, U. C. Berkeley, U. C. Santa Cruz and U of Minnesota between February 2000 and May 2000. I would like to thank Professor R. Schoen and Professor S.-T. Yau for their constant encouragement and invaluable advice. I also have benefitted greatly from the many discussions I had with Professor G. Huisken, Professor L. Simon and Professor B. White.

2. Evolution equations of parallel forms

Let $F : \Sigma^2 \to M^4$ be an isometric immersion of an orientable surface into a four-dimensional Riemannian manifold. We fixed an orientation on Σ. The restriction of the tangent bundle of M to Σ splits as the direct sum of the tangent bundle of Σ and the normal bundle:

$$TM|_\Sigma = T\Sigma \oplus N\Sigma.$$

The Levi-Civita connection on M induces a connection on $T\Sigma$. We denote the connection on M by $\overline{\nabla}$ and the induced connection on $T\Sigma$ by ∇. Therefore,

$$\nabla_X Y = (\overline{\nabla}_X Y)^T$$

for any tangent vector fields X, Y. Here $(\cdot)^T$ denotes the projection from TM onto $T\Sigma$ and $(\cdot)^N$ shall denote the projection onto $N\Sigma$.

The second fundamental form $A : T\Sigma \times T\Sigma \mapsto N\Sigma$ is defined by $A(X,Y) = (\overline{\nabla}_X Y)^N$. We also define $B : T\Sigma \times N\Sigma \mapsto T\Sigma$ by $B(X,N) = (\overline{\nabla}_X N)^T$. The relation between A and B is

$$\langle A(X,Y), N\rangle = -\langle Y, B(X,N)\rangle.$$

Note that we have identified $X \in T\Sigma$ with $F_*(X) \in TM$.

Fix a point $p \in \Sigma$. Let $\{x^i\}$ be a normal coordinate system for Σ at p and $\{y^A\}$ a normal coordinate system for M at $F(p)$. We denote $\frac{\partial}{\partial x^i}$ by ∂_i and identify it with $\frac{\partial F}{\partial x_i}$. The induced metric on Σ is given by $g_{kl} = \langle \partial_k, \partial_l \rangle$. The mean curvature vector along Σ is the trace of A, i.e., $H = g^{kl} A(\partial_k, \partial_l)$.

Let $\overline{\omega}$ be a parallel two form on M and $\omega = F^* \overline{\omega}$ be the pull-back of ω on Σ. We first compute the rough Laplacian of ω on Σ:

$$\Delta \omega = g^{kl} \nabla_{\partial_k} \nabla_{\partial_l} \omega.$$

Lemma 2.1.

$$\begin{aligned}(\Delta\omega)(X,Y) =& \overline{\omega}((\overline{\nabla}_X H)^N, Y) - \overline{\omega}((\overline{\nabla}_Y H)^N, X) \\ &- g^{kl}\overline{\omega}((K(\partial_k, X)\partial_l)^N, Y) \\ &+ g^{kl}\overline{\omega}((K(\partial_k, Y)\partial_l)^N, X) \\ &+ g^{kl}\overline{\omega}(B(\partial_k, A(\partial_l, X)), Y) \\ &- g^{kl}\overline{\omega}(B(\partial_k, A(\partial_l, Y)), X) \\ &+ 2g^{kl}\overline{\omega}(A(\partial_k, X), A(\partial_l, Y))\end{aligned} \tag{2.1}$$

where $K(X,Y)Z = -\overline{\nabla}_X \overline{\nabla}_Y Z + \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z$ is the curvature operator of M. Notice that $\langle K(X,Y)X, Y\rangle > 0$ if M has positive sectional curvature.

Proof. Since both sides are tensors, we calculate at the point p using a normal coordinate system. Therefore $g_{kl} = \delta_{kl}$ and all connection terms vanish. Now

$$(\Delta\omega)(\partial_i, \partial_j) = \partial_k[\partial_k(\omega(\partial_i, \partial_j)) - \omega(\nabla_{\partial_k}\partial_i, \partial_j) - \omega(\partial_i, \nabla_{\partial_k}\partial_j)].$$

The term in the bracket is

$$\begin{aligned}\partial_k(\omega(\partial_i, \partial_j)) - &\omega(\nabla_{\partial_k}\partial_i, \partial_j) - \omega(\partial_i, \nabla_{\partial_k}\partial_j) \\ &= (\overline{\nabla}_{\partial_k}\overline{\omega})(\partial_i, \partial_j) + \overline{\omega}(\overline{\nabla}_{\partial_k}\partial_i, \partial_j) + \overline{\omega}(\partial_i, \overline{\nabla}_{\partial_k}\partial_j) \\ &\quad - \overline{\omega}(\nabla_{\partial_k}\partial_i, \partial_j) - \overline{\omega}(\partial_i, \nabla_{\partial_k}\partial_j) \\ &= \overline{\omega}(A(\partial_k, \partial_i), \partial_j) + \overline{\omega}(\partial_i, A(\partial_k, \partial_j))\end{aligned}$$

where we have used the fact that $\overline{\omega}$ is parallel and $\overline{\nabla}_{\partial_k}\partial_i - \nabla_{\partial_k}\partial_i = A(\partial_k, \partial_i)$.

Therefore

$$\Delta\omega(\partial_i, \partial_j) = \partial_k[\overline{\omega}(A(\partial_k, \partial_i), \partial_j) + \overline{\omega}(\partial_i, A(\partial_k, \partial_j))]. \tag{2.2}$$

Use Leibnitz rule and the parallelity of $\overline{\omega}$ again.

$$\begin{aligned}
&\partial_k(\overline{\omega}(A(\partial_k,\partial_i),\partial_j)\\
&\quad=\overline{\omega}(\overline{\nabla}_{\partial_k}A(\partial_k,\partial_i),\partial_j)+\overline{\omega}(A(\partial_k,\partial_i),\overline{\nabla}_{\partial_k}\partial_j)\\
&\quad=\overline{\omega}((\overline{\nabla}_{\partial_k}A(\partial_k,\partial_i))^T+(\overline{\nabla}_{\partial_k}A(\partial_k,\partial_i))^N,\partial_j)\\
&\qquad+\overline{\omega}(A(\partial_k,\partial_i),\nabla_{\partial_k}\partial_j+A(\partial_k,\partial_j))\\
&\quad=\overline{\omega}(B(\partial_k,A(\partial_k,\partial_i)),\partial_j)+\overline{\omega}(A(\partial_k,\partial_i),A(\partial_k,\partial_j))\\
&\qquad+\overline{\omega}((\overline{\nabla}_{\partial_k}A(\partial_k,\partial_i))^N,\partial_j)
\end{aligned}$$

where we have used $\nabla_{\partial_k}\partial_j=0$ at the point p in normal coordinates.

$$\begin{aligned}
&\overline{\omega}((\overline{\nabla}_{\partial_k}A(\partial_k,\partial_i))^N,\partial_j)\\
&\quad=\overline{\omega}((\overline{\nabla}_{\partial_k}\overline{\nabla}_{\partial_i}\partial_k)^N,\partial_j)-\overline{\omega}((\overline{\nabla}_{\partial_k}\nabla_{\partial_i}\partial_k)^N,\partial_j)\\
&\quad=\overline{\omega}((-K(\partial_k,\partial_i)\partial_k,+\overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_k}\partial_k)^N,\partial_j)-\overline{\omega}(B(\partial_k,\nabla_{\partial_i}\partial_k),\partial_j)\\
&\quad=-\overline{\omega}((K(\partial_k,\partial_i)\partial_k)^N,\partial_j)+\overline{\omega}((\overline{\nabla}_{\partial_i}H)^N+(\overline{\nabla}_{\partial_i}\nabla_{\partial_k}\partial_k)^N,\partial_j)\\
&\quad=-\overline{\omega}((K(\partial_k,\partial_i)\partial_k)^N,\partial_j)+\overline{\omega}((\overline{\nabla}_{\partial_i}H)^N,\partial_j)\\
&\qquad+\overline{\omega}(B(\partial_i,\nabla_{\partial_k}\partial_k)),\partial_j).
\end{aligned}$$

The last term vanishes in normal coordinates.

Thus we have proved

$$\begin{aligned}
\partial_k(\overline{\omega}(A(\partial_k,\partial_i),\partial_j))&=\overline{\omega}(B(\partial_k,A(\partial_k,\partial_i)),\partial_j)+\overline{\omega}(A(\partial_k,\partial_i),A(\partial_k,\partial_j))\\
&\quad-\overline{\omega}((K(\partial_k,\partial_i)\partial_k)^N,\partial_j)+\overline{\omega}((\overline{\nabla}_{\partial_i}H)^N,\partial_j).
\end{aligned}$$

Plug this equation back into Equation (2.2) and anti-symmetrize i, j and the lemma is proved. q.e.d.

Let's represent the fixed orientation on Σ by a two-form $d\mu$. Let $F:\Sigma\times[0,T)\to M$ be the mean curvature flow of Σ. The immersion F_t induces a pull-back metric g_t on Σ. The volume form of g_t is denoted by $d\mu_t=\sqrt{\det g_t}\,d\mu$.

Now we consider the evolution equation of $\omega_t=F_t^*(\overline{\omega})$. This is a family of time-dependent two forms on the fixed surface Σ. Let the one-form α_t be defined by $\alpha_t(X)=\overline{\omega}(H_t,X)$.

Lemma 2.2. *Along the mean curvature flow*

$$\frac{d}{dt}\omega_t=d\alpha_t.$$

For any vector field $X,Y\in T\Sigma$,

$$\begin{aligned}
\frac{d}{dt}\omega_t(X,Y)&=\overline{\omega}((\overline{\nabla}_XH)^N,Y)+\overline{\omega}(X,(\overline{\nabla}_YH)^N)\\
&\quad+\overline{\omega}(B(X,H),Y)+\overline{\omega}(X,B(Y,H)).
\end{aligned}$$

Proof.

$$\frac{d}{dt}\omega_t(\partial_i,\partial_j)=\overline{\omega}(\overline{\nabla}_H\partial_i,\partial_j)+\overline{\omega}(\partial_i,\overline{\nabla}_H\partial_j)=\overline{\omega}(\overline{\nabla}_{\partial_i}H,\partial_j)+\overline{\omega}(\partial_i,\overline{\nabla}_{\partial_j}H).$$

By definition $\overline{\nabla}_{\partial_i}H=(\overline{\nabla}_{\partial_i}H)^N+B(\partial_i,H)$. On the other hand,

$$d\alpha_t(\partial_i,\partial_j)=\partial_i(\overline{\omega}(H,\partial_j))-\partial_j(\overline{\omega}(H,\partial_i))=\overline{\omega}(\overline{\nabla}_{\partial_i}H,\partial_j)-\overline{\omega}(\overline{\nabla}_{\partial_j}H,\partial_i).$$

q.e.d.

The volume form $d\mu_t$ determines a Hodge operator $*_t$. Therefore $*_t\omega_t$ becomes a time-dependent function on Σ.

Proposition 2.1. *Let $\overline{\omega}$ be a parallel two-form on M. $F_t:\Sigma\mapsto M$ be the t slice of a mean curvature flow and $\omega_t=F_t^*(\overline{\omega})$ be the pull-back form on Σ. Then $\eta_t=*_t\omega_t$ satisfies the following parabolic equation:*

$$\begin{aligned}\frac{d}{dt}\eta_t=&(\Delta_t\eta_t)+|A|^2\eta_t\\&-2\overline{\omega}(A(e_k,e_1),A(e_k,e_2))+\overline{\omega}((K(e_k,e_1)e_k)^N,e_2)\\&-\overline{\omega}((K(e_k,e_2)e_k)^N,e_1)\end{aligned}$$

where $|A|$ is the norm of the second fundamental form,

$$|A|^2=g^{ij}g^{kl}\langle A(\partial_i,\partial_k),A(\partial_j,\partial_l)\rangle$$

and $\{e_1,e_2\}$ any orthonormal basis with respect to g_t.

Proof. Combining the previous two lemmas, we get

$$\begin{aligned}\frac{d}{dt}\omega_t(X,Y)=&(\Delta_t\omega_t)(X,Y)+\overline{\omega}(B(X,H),Y)\\&+\overline{\omega}(X,B(Y,H))+g^{kl}\overline{\omega}((K(\partial_k,X)\partial_l)^N,Y)\\&-g^{kl}\overline{\omega}((K(\partial_k,Y)\partial_l)^N,X)\\&-g^{kl}\overline{\omega}(B(\partial_k,A(\partial_l,X)),Y)\\&+g^{kl}\overline{\omega}(B(\partial_k,A(\partial_l,Y)),X)\\&-2g^{kl}\overline{\omega}(A(\partial_k,X),A(\partial_l,Y)).\end{aligned}\tag{2.3}$$

Now $*_t\omega_t=\frac{\omega(\partial_1,\partial_2)}{\sqrt{\det g_t}}$ where $\{\partial_1,\partial_2\}$ is a fixed coordinate system on Σ and $\det g_t$ is the determinant of $(g_t)_{ij}=\langle(F_t)_*\partial_i,(F_t)_*\partial_j\rangle$.

It is easy to compute

$$\frac{d}{dt}\sqrt{\det g_t}=-|H|^2\sqrt{\det g_t}$$

where $|H|$ is the norm of the mean curvature vector.

Thus

$$\frac{d}{dt} *_t \omega_t = \frac{1}{\sqrt{\det g_t}}\frac{d}{dt}\omega_t(\partial_1, \partial_2) + |H|^2 *_t \omega_t.$$

Now we use Equation (2.3) with $X = \partial_1$ and $Y = \partial_2$.

The first term is $\frac{1}{\sqrt{\det g_t}}(\Delta_t\omega_t)(\partial_1, \partial_2) = *_t\Delta_t\omega_t = \Delta_t *_t \omega_t$ because the Hodge $*_t$ operator is parallel.

For other terms we can take any orthonormal basis $\{e_1, e_2\}$ with respect to the metric g_t to calculate.

It is not hard to see

$$\begin{aligned}
&\overline{\omega}(B(e_1, H), e_2) - \overline{\omega}(B(e_2, H), e_1) \\
&\qquad = *_t\omega_t(\langle B(e_1, H), e_1\rangle + \langle B(e_2, H), e_2\rangle) \\
&\qquad = - *_t \omega_t(\langle A(e_1, e_1), H\rangle + \langle A(e_2, e_2), H\rangle) \\
&\qquad = - *_t \omega_t |H|^2.
\end{aligned}$$

Likewise,

$$\begin{aligned}
&\overline{\omega}(B(e_k, A(e_k, e_1)), e_2) - \overline{\omega}(B(e_k, A(e_k, e_2)), e_1) \\
&\quad = *_t\omega_t(\langle B(e_k, A(e_k, e_1)), e_1\rangle + \langle B(e_k, A(e_k, e_2)), e_2\rangle) \\
&\quad = - *_t \omega_t(\langle A(e_k, e_1), A(e_k, e_1)\rangle + \langle A(e_k, e_2), A(e_k, e_2)\rangle).
\end{aligned}$$

q.e.d.

3. Calibrating two-forms in four-dimensional spaces

Let $V \simeq \mathbb{R}^4$ be an inner product space and $\alpha \in \wedge^2 V^*$ a two form. We shall use the inner product to identify V and V^* and this induces inner product on all $\wedge^k V^*$. First let's recall the definition of comass of α,

$$\text{comass}(\alpha) = \max_{x \in G(2,V)} \alpha(x)$$

where $G(2, V)$ is the Grassmanian of all two-planes in V. $G(2, V)$ can be described by

$$G(2, V) = \{x \in \wedge^2 V, x \wedge x = 0 \,\text{and}\, |x|^2 = 1\}.$$

Now we fix an orientation $\nu \in \wedge^4 V^*$ and normalize so that $|\nu| = 1$. Given any orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for V such that

$\nu(e_1, e_2, e_3, e_4) = 1$, the following two-forms give an orthonormal basis for $\wedge^2 V^*$.

$$\begin{array}{ll} \alpha_1 = \frac{1}{\sqrt{2}}(e_1^* \wedge e_2^* + e_3^* \wedge e_4^*) & \beta_1 = \frac{1}{\sqrt{2}}(e_1^* \wedge e_2^* - e_3^* \wedge e_4^*) \\ \alpha_2 = \frac{1}{\sqrt{2}}(e_1^* \wedge e_3^* - e_2^* \wedge e_4^*) & \beta_2 = \frac{1}{\sqrt{2}}(e_1^* \wedge e_3^* + e_2^* \wedge e_4^*) \\ \alpha_3 = \frac{1}{\sqrt{2}}(e_1^* \wedge e_4^* + e_2^* \wedge e_3^*) & \beta_3 = \frac{1}{\sqrt{2}}(e_1^* \wedge e_4^* - e_2^* \wedge e_3^*). \end{array}$$

These forms serve as coordinate functions on $G(2, V)$, under the identification

$$x \to (\alpha_i(x), \beta_i(x)).$$

An element x in $G(2, V)$ satisfies $\sum_i (\alpha_i(x))^2 = \sum_i (\beta_i(x))^2 = \frac{1}{2}$. Therefore $G(2, V) \backsimeq S^2\left(\frac{1}{\sqrt{2}}\right) \times S^2\left(\frac{1}{\sqrt{2}}\right)$.

Now for any given $\alpha \in \wedge^2 V^*$. We identify α with an element K in $\mathrm{End}(V)$ by $\alpha(X, Y) = \langle K(X), Y\rangle$. Since $\sqrt{-1}\, K$ is Hermitian symmetric and purely imaginary, it has real eigenvalues $\pm\lambda_1, \pm\lambda_2$. We can require $\alpha \wedge \alpha = \lambda_1\lambda_2\nu$. $\lambda_1\lambda_2$ is actually the Pfaffian of α and $\det K = (\lambda_1\lambda_2)^2$. A form is self-dual (anti-self-dual) if $\lambda_1\lambda_2 = 1(-1)$.

Lemma 3.1.

$$\mathrm{comass}(\alpha) = \max\{|\lambda_1|, |\lambda_2|\}.$$

Proof. We can find an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ with $\nu(e_1, e_2, e_3, e_4) = 1$ such that $\alpha = \lambda_1 e_1^* \wedge e_2^* + \lambda_2 e_3^* \wedge e_4^*$. In terms of the self-dual and anti-self-dual bases associated with $\{e_1, e_2, e_3, e_4\}$.

$$\alpha = \frac{1}{\sqrt{2}}(\lambda_1 + \lambda_2)\alpha_1 + \frac{1}{\sqrt{2}}(\lambda_1 - \lambda_2)\beta_1.$$

Therefore

$$\begin{aligned} \alpha(x) &= \frac{1}{\sqrt{2}}(\lambda_1 + \lambda_2)\alpha_1(x) + \frac{1}{\sqrt{2}}(\lambda_1 - \lambda_2)\beta_1(x) \\ &\leq \frac{1}{2}(|\lambda_1 + \lambda_2| + |\lambda_1 - \lambda_2|) \\ &= \max\{\lambda_1, \lambda_2\}. \end{aligned}$$

We notice that if $|\lambda_1| \neq |\lambda_2|$, a unique plane is calibrated by α. However if $|\lambda_1| = |\lambda_2|$, then a two-dimensional family of planes in $G(2, V)$ are calibrated by α. q.e.d.

Lemma 3.2. *A self-dual or anti-self-dual calibrating form α can be written as $\alpha(\cdot,\cdot) = \langle K(\cdot),\cdot\rangle$ with $K \in O(4, V)$, the orthogonal group.*

Proof. If α is calibrating and self-dual or anti-self-dual, then $\max\{\lambda_1, \lambda_2\} = 1$ and $\lambda_1\lambda_2 = \pm 1$, therefore $\lambda_1 = \pm 1$ and it is not hard to see that K is an isometry.

On the other hand, if α is induced by an isometry J, then $\det J = \pm 1$, therefore $\lambda_1\lambda_2 = \pm 1$ and α is self-dual or anti-self-dual. q.e.d.

Proposition 3.1. *Let (x, μ) be an oriented two-plane in V. Let α be a self-dual calibrating form and β be a anti-self-dual calibrating form. Then there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for V with $\{e_1, e_2\}$ a basis for x such that $\mu(e_1, e_2) > 0$, $\nu(e_1, e_2, e_3, e_4) > 0$, $\alpha(e_A, e_B), A, B = 1\cdots 4$ is of the form*

$$\begin{pmatrix} 0 & \eta_1 & \zeta_1 & 0 \\ -\eta_1 & 0 & 0 & -\zeta_1 \\ -\zeta_1 & 0 & 0 & \eta_1 \\ 0 & \zeta_1 & -\eta_1 & 0 \end{pmatrix} \tag{3.1}$$

where $\eta_1 = \alpha(e_1, e_2)$, $\zeta_1^2 + \eta_1^2 = 1$, and $\beta(e_A, e_B)$ is of the form.

$$\begin{pmatrix} 0 & \eta_2 & \zeta_2 & 0 \\ -\eta_2 & 0 & 0 & \zeta_2 \\ -\zeta_2 & 0 & 0 & -\eta_2 \\ 0 & -\zeta_2 & \eta_2 & 0 \end{pmatrix} \tag{3.2}$$

where $\eta_2 = \beta(e_1, e_2)$ and $\eta_2^2 + \zeta_2^2 = 1$.

Proof. Let K, L be the elements in $\mathrm{End}(V)$ corresponding to α and β. If $\eta_1 \neq \pm 1$, we take any orthonormal basis $\{e_1, e_2\}$ with $\mu(e_1, e_2) > 0$. Notice that $(Ke_1)^T = \eta_1 e_2$. Let

$$e_3 = \frac{1}{\sqrt{1-\eta_1^2}}(Ke_1 - \eta_1 e_2)$$
$$e_4 = \frac{-1}{\sqrt{1-\eta_1^2}}(Ke_2 + \eta_1 e_1).$$

Therefore $\alpha_{A,B}$ is of the required form. If $\eta_1 = \pm 1$, then any $\{e_1, e_2, e_3, e_4\}$ compatible with μ and ν works.

It is not hard to check that K and L as elements in $\mathrm{End}(V)$ commute and KL is a self-adjoint operator. Therefore we can rotate $\{e_1, e_2\}$ to get a new basis so that $\langle KLe_1, e_2\rangle = 0$.

This implies

$$\begin{aligned}\langle Le_1, e_4\rangle &= \frac{-1}{\sqrt{1-\eta_1^2}}\langle Le_1, Ke_2 + \eta_1 e_1\rangle \\ &= \frac{1}{\sqrt{1-\eta_1^2}}\langle KLe_1, e_2\rangle = 0.\end{aligned}$$

Likewise $\langle Le_2, e_3\rangle = 0$. That $\langle Le_1, e_3\rangle = -\langle Le_2, e_4\rangle$ follows from the fact that β is anti-self-dual. q.e.d.

Finally, we make a remark about $\alpha+\beta$. In the above basis $\alpha+\beta$ is of the form

$$\begin{pmatrix} 0 & \eta_1+\eta_2 & \zeta_1+\zeta_2 & 0 \\ -\eta_1-\eta_2 & 0 & 0 & -\zeta_1+\zeta_2 \\ -\zeta_1-\zeta_2 & 0 & 0 & \eta_1-\eta_2 \\ 0 & \zeta_1-\zeta_2 & -\eta_1+\eta_2 & 0 \end{pmatrix}. \tag{3.3}$$

If the eigenvalues of $\sqrt{-1}(\alpha+\beta)$ are $\pm\lambda_1$ and $\pm\lambda_2$, then it is not hard to compute that $\lambda_1^2\lambda_2^2 = ((\eta_1+\eta_2)(\eta_1-\eta_2)-(\zeta_1+\zeta_2)(-\zeta_1+\zeta_2))^2 = 0$ and $\lambda_1^2+\lambda_2^2 = (\eta_1+\eta_2)^2+(\zeta_1+\zeta_2)^2+(\eta_1-\eta_2)^2+(\zeta_1-\zeta_2)^2 = 4$. Therefore $\frac{1}{2}(\alpha+\beta)$ is a calibrating form and calibrates a unique two-plane.

4. Surfaces in Kähler manifolds

In this section, we assume $\overline{\omega}$ is a parallel self-dual calibrating two form and $\overline{\omega}(X,Y) = \langle J(X), Y\rangle$. J is then a parallel almost complex structure. M is therefore a Kähler manifold with Kähler form $\overline{\omega}$.

We shall compute the equation of $\eta_t = *_t\omega_t$ along the mean curvature flow.

The following Lemma is well-known.

Lemma 4.1. *Let $K(\cdot,\cdot)$ be the curvature operator of M and* $\mathrm{Ric}(\cdot,\cdot)$ *be the Ricci tensor of M. In terms of any orthonormal basis $\{e_1,e_2,e_3,e_4\}$, the Ricci form is*

$$\mathrm{Ric}(JX,Y) = \frac{1}{2}K(X,Y,e_A,J(e_A)).$$

Proof. This is seen by the following calculation:

$$\begin{aligned}&K(JX,e_A,Y,e_A)\\ &\qquad = K(JX,e_A,J(Y),J(e_A))\\ &\qquad = -K(JX,JY,J(e_A),e_A) - K(JX,J(e_A),e_A,J(Y))\\ &\qquad = K(X,Y,e_A,J(e_A)) - K(JX,J(e_A),Y,J(e_A)).\end{aligned}$$

Now $K(JX, e_A, Y, e_A) = K(JX, J(e_A), Y, J(e_A))$ since $\{J(e_A)\}$ is also an orthonormal basis. q.e.d.

Let $F : \Sigma \to M$ be an isometric immersion. Σ is equipped with a fixed orientation $d\mu$. By Proposition 3.1, for any point $p \in \Sigma$ it is possible to choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for T_pM such that $d\mu(e_1, e_2) > 0$ and

$$\begin{aligned}\overline{\omega}^2(e_1, e_2, e_3, e_4) = \overline{\omega}(e_1, e_2)\overline{\omega}(e_3, e_4) - \overline{\omega}(e_1, e_3)\overline{\omega}(e_2, e_4)\\ + \overline{\omega}(e_1, e_4)\overline{\omega}(e_2, e_3) > 0\end{aligned}$$

and such that $\overline{\omega}_{A,B} = \overline{\omega}(e_A, e_B), A, B = 1 \cdots 4$ is of the form.

$$\begin{pmatrix} 0 & \eta & \sqrt{1-\eta^2} & 0 \\ -\eta & 0 & 0 & -\sqrt{1-\eta^2} \\ -\sqrt{1-\eta^2} & 0 & 0 & \eta \\ 0 & \sqrt{1-\eta^2} & -\eta & 0 \end{pmatrix} \tag{4.1}$$

where $\eta = \omega(e_1, e_2) = *\omega$.

We first use this basis to calculate the curvature term in Proposition (2.1).

Proposition 4.1. *Let* $\mathrm{Ric}(\cdot, \cdot)$ *be the Ricci tensor of* M. $\overline{\omega}$ *a parallel Kähler form. Then* $\eta = *_t\omega_t$ *satisfies the following equation:*

(4.2)
$$\frac{d}{dt}\eta = \Delta\eta + \eta\left[(h_{31k} - h_{42k})^2 + (h_{32k} + h_{41k})^2\right] + (1 - \eta^2)\,\mathrm{Ric}(Je_1, e_2)$$

where $\{e_1, e_2, e_3, e_4\}$ *is any orthonormal basis for* T_pM *such that* $\{e_1, e_2\}$ *forms an orthonormal basis for* $T\Sigma_t$*,* $d\mu(e_1, e_2) > 0$ *and* $\overline{\omega}^2(e_1, e_2, e_3, e_4) > 0$*.* $A(e_i, e_j) = h_{3ij}e_3 + h_{4ij}e_4$ *is the second fundamental form.*

Remark 4.1. Notice that the term $(h_{31k} - h_{42k})^2 + (h_{32k} + h_{41k})^2$ depends only on the orientation of $\{e_1, e_2, e_3, e_4\}$ but not on the particular orthonormal basis we choose.

Proof. First we show

(4.3)
$$\overline{\omega}((K(e_k, e_1)e_k)^N, e_2) - \overline{\omega}((K(e_k, e_2)e_k)^N, e_1) = (1 - \eta^2)\,\mathrm{Ric}(Je_1, e_2).$$

By definition,

$$\begin{aligned}\overline{\omega}((K(e_k, e_1)e_k)^N, e_2) - \overline{\omega}((K(e_k, e_2)e_k)^N, e_1)\\ = -\langle (Je_2)^N, K(e_k, e_1)e_k\rangle + \langle (Je_1)^N, K(e_k, e_2)e_k\rangle.\end{aligned}$$

Therefore when $\eta = \pm 1$, Equation (4.3) is obvious. Therefore we may assume $\eta \neq \pm 1$ and apply the basis in Equation (4.1) and get

$$\begin{aligned}&\sqrt{1-\eta^2}(K(e_k, e_2, e_k, e_3) + K(e_k, e_1, e_k, e_4))\\ &\quad = \sqrt{1-\eta^2}(K(e_1, e_2, e_1, e_3) + K(e_2, e_1, e_2, e_4)).\end{aligned}$$

By the previous lemma,

$$\begin{aligned}\mathrm{Ric}(JX, Y) &= \frac{1}{2} J_{AB} K(X, Y, e_A, e_B)\\ &= \eta(K(X, Y, e_1, e_2) + K(X, Y, e_3, e_4))\\ &\quad + \sqrt{1-\eta^2}(K(X, Y, e_1, e_3) - K(X, Y, e_2, e_4)).\end{aligned}$$

Since J is parallel and isometry, the curvature tensor is J invariant, therefore we have

$$K(X, Y, e_1, e_2) = K(X, Y, J(e_1), J(e_2)).$$

Use (4.1) again, this is the same as

$$\begin{aligned}&(1-\eta^2)(K(X, Y, e_1, e_2) + K(X, Y, e_3, e_4))\\ &\quad = \eta\sqrt{1-\eta^2}(K(X, Y, e_1, e_3) - K(X, Y, e_2, e_4)).\end{aligned}$$

Therefore

$$\mathrm{Ric}(JX, Y) = \frac{1}{\sqrt{1-\eta^2}}(K(X, Y, e_1, e_3) - K(X, Y, e_2, e_4)).$$

Equation (4.3) now follows by substituting $X = e_1, Y = e_2$.

We use the basis in Equation (4.1) to calculate the rest terms in Proposition 2.1.

$$\begin{aligned}&\eta|A|^2 - 2\overline{\omega}(A(e_k, e_1), A(e_k, e_2))\\ &\quad = \eta(|A|^2 - 2h_{31k}h_{42k} + 2h_{41k}h_{32k}).\end{aligned}$$

Equation (4.2) follows by completing squares. q.e.d.

Remark 4.2. When M is a Kähler manifold with Kähler form $\overline{\omega}$ and almost complex structure J. The second fundamental form of a holomorphic submanifold has the symmetry $h_{31k} = h_{42k}, h_{41k} = -h_{32k}$.

5. Asymptotics of singularities

In this section, we study the asymptotic behavior of singularities of the mean curvature flow. In particular, we show that no type I singularity will occur in the mean curvature flow of symplectic surface in a four-dimensional Kähler-Einstein manifold. Techniques involved are blow-up analysis and monotonicity formula of backward heat kernel.

The following lemma says singularity forms only when the second fundamental form blows up.

Lemma 5.1. *Given any mean curvature flow $F : \Sigma \times [0, t_0) \to M$, suppose $\sup_{t\in[0,t_0)} \sup_{x\in\Sigma} |A|(x,t)$ is bounded where $|A|(x,t)$ is the norm of the second fundamental form for $F_t(\Sigma)$ at $F_t(x)$. Then F can be extended to $\Sigma \times [0, \bar{t}_0)$ for some $\bar{t}_0 > t_0$.*

Proof. It can be shown that all higher covariant derivatives of the second fundamental form are uniformly bounded. For the detail see [2] for the hypersurface case. q.e.d.

Since the study of singularities is local, it is more convenient to adopt an unparametrized definition of mean curvature flow introduced in [12]. Let M be an m-dimensional Riemannian manifold of bounded geometry. An immersed smooth submanifold $\mathcal{S} \subset M \times \mathbb{R}$ is a smooth flow if the function $\tau : M \times \mathbb{R} \to \mathbb{R}$, $\tau(y,t) = t$ has no critical points in $\mathcal{S}$. $\mathcal{S}_t = \mathcal{S} \cap M \times \{t\}$ is called the t-slice of $\mathcal{S}$. At each point $(y,t) \in \mathcal{S}$, the normal velocity $v(y,t)$ is the unique vector that satisfies v is normal to $\mathcal{S}_t$ and $v + \frac{\partial}{\partial t}$ is tangent to $\mathcal{S}$. $H(y,t)$ is the mean curvature vector of $\mathcal{S}_t$ at (y,t). We allow M and $\mathcal{S}_t$ to have boundary. In fact, all unparametrized flow considered in this article is of the form $\cup_{t\in[0,t_0)}(F_t(\Sigma) \cap B) \times \{t\}$, where F is a parameterized mean curvature flow of a compact manifold Σ without boundary and B is an neighborhood of y in a complete Riemannian manifold. Therefore $\partial\mathcal{S}_t \subset \partial M$.

Definition 5.1. A smooth flow $\mathcal{S}$ is called a (unparametrized) mean curvature flow if

$$v(y,t) = H(y,t)$$

at each point $(y,t) \in \mathcal{S}$.

Let (y_0, t_0) be an interior point in $M \times \mathbb{R}$. When M is the Euclidean space, in [3] Huisken introduces the backward heat kernel to study the asymptotic behavior near singular points. Recall the (n-dimensional) backward heat kernel ρ_{y_0,t_0} at (y_0, t_0):

$$\rho_{y_0,t_0}(y,t) = \frac{1}{(4\pi(t_0-t))^{\frac{n}{2}}} \exp\left(\frac{-|y-y_0|^2}{4(t_0-t)}\right). \tag{5.1}$$

The monotonicity formula of Huisken asserts for $t < t_0$

$$\frac{d}{dt}\int \rho_{y_0,t_0} d\mu_t \leq 0.$$

For general Riemannian manifold M, following [11], we isometrically embed M into $\mathbb{R}^N$. The mean curvature flow of Σ in M now reads.

$$\frac{d}{dt}F = H = \overline{H} + E$$

where F is the coordinate function in $\mathbb{R}^N$, H is the mean curvature vector of Σ in M, $\overline{H}$ is the mean curvature vector of Σ in $\mathbb{R}^N$, and

$$E = \sum_i \overline{A}(e_i, e_i).$$

Here $\overline{A}$ denotes the second fundamental form of M in $\mathbb{R}^N$ and $\{e_i\}$ is an orthonormal basis for $T\Sigma_t$.

In the general case $\int \rho_{y_0,t_0} d\mu_t$ is no longer decreasing, however the following is still true:

Proposition 5.1. *Let $\mathcal{S} \subset M \times \mathbb{R}$ be a mean curvature flow such that $\partial\mathcal{S}_t \subset \partial M$. We fix an isometric embedding $M \hookrightarrow \mathbb{R}^N$ and let ρ_{y_0,t_0} be the (n-dimensional) backward heat kernel at (y_0, t_0). Then the limit*

$$\lim_{t\to t_0} \int \rho_{y_0,t_0} d\mu_t$$

exists, where $d\mu_t$ is the Radon measure associated with $\mathcal{S}_t \subset M$.

Proof. See Proposition 11 in [11]. q.e.d.

The limit is called the Gaussian density of $\mathcal{S}$ at (y_0, t_0) in [12]. The Gaussian density can be used to detect singularities of mean curvature flow. The following theorem of White in [12] is a parabolic analogue of Allard's regularity theorem.

Theorem 5.1. *There is an $\epsilon > 0$ such that whenever*

$$\lim_{t\to t_0} \int \rho_{y_0,t_0} d\mu_t < 1 + \epsilon,$$

it can concluded that (y_0, t_0) is a regular point of $\mathcal{S}$.

A regular point is a point where the second fundamental form is locally bounded in Hölder norm.

To study singularity, we consider the parabolic blow-up near a possible singular point. Let $F : \Sigma \times [0, t_0) \to M \hookrightarrow \mathbb{R}^N$ be a parameterized mean curvature flow. Let B be a ball about y_0 of radius r in $\mathbb{R}^N$. Take $\mathcal{S} = \cup_{t\in[0,t_0)}(F_t(\Sigma) \cap B) \times \{t\}$, then $\mathcal{S}$ is an unparametrized mean curvature flow in B.

For any $\lambda > 1$, the parabolic dilation D_λ at (y_0, t_0) is defined by

$$\begin{aligned} D_\lambda : \mathbb{R}^N \times [0, t_0) &\to \mathbb{R}^N \times [-\lambda^2 t_0, 0) \\ (y, t) &\to (\lambda(y - y_0), \lambda^2(t - t_0)). \end{aligned} \tag{5.2}$$

For any s, $-\lambda^2 t_0 \le s < 0$, the two slices $\mathcal{S}^\lambda_s$ and $\mathcal{S}_{t_0+\frac{s}{\lambda^2}}$ can be identified and $d\mu^\lambda_s = \lambda^n d\mu_t$.

It is not hard to check that if we denote $F^\lambda_s(x) = \lambda(F_t(x) - y_0)$ for $s = \lambda^2(t - t_0)$, then

$$\rho_{0,0}(F^\lambda_s(x), s) = \rho_{0,0}(\lambda(F_t(x) - y_0), \lambda^2(t - t_0)) = \frac{1}{\lambda^n}\rho_{y_0,t_0}(F_t(x), t).$$

Therefore

$$\int \rho_{y_0,t_0} d\mu_t = \int \rho_{0,0} d\mu^\lambda_s$$

is invariant under the parabolic dilation.

The singularity of $\mathcal{S}$ near (y_0, t_0) is reflected in the asymptotic behavior of $\mathcal{S}^\lambda$ as $\lambda \to \infty$.

Take any sequence $\lambda_i \to \infty$, it can be proved as in [4] and [11] that a subsequence of $\mathcal{S}^{\lambda_i}$ converges to a Brakke flow $\mathcal{S}^\infty \subset \mathbb{R}^N \times (-\infty, 0)$. $\mathcal{S}^\infty$ is called a tangent flow of $\mathcal{S}$ at (y_0, t_0).

Now we state and prove the main proposition in this section.

Proposition 5.2. *If $F : \Sigma \times [0, t_0) \to M \hookrightarrow \mathbb{R}^N$ is a mean curvature flow of an orientable surface in a (real) four dimensional Kähler manifold M. Assume the second fundamental form of $M \hookrightarrow \mathbb{R}^N$ is bounded. Let $\omega(\cdot, \cdot) = \langle J(\cdot), \cdot\rangle$ be a Kähler form on M. If there exist $\delta, C > 0$ such that $\eta_t = *\omega_t > \delta$ on $F_t(\Sigma)$ for $t \in [0, t_0)$ and such that $|A|^2 \le \frac{C}{t_0 - t}$, then F can be extended to $\Sigma \times [0, \bar{t}_0)$ for some $\bar{t}_0 > t_0$.*

Proof. For $y_0 \in M$, we shall consider the blow up of the mean curvature flow at (y_0, t_0). Let B be a ball of radius r about y_0 in $\mathbb{R}^N$

and ψ be a cut-off function supported in B so that $\psi \equiv 1$ in the ball of radius $\frac{r}{2}$ about y_0. We assume

$$|\overline{\nabla}\psi| + |\overline{\nabla}\overline{\nabla}\psi| \leq C$$

where $\overline{\nabla}$ is the covariant derivative on $\mathbb{R}^N$. Recall the equation for η is

$$\frac{d}{dt}\eta = \Delta\eta + \eta\left[(h_{31k} - h_{42k})^2 + (h_{32k} + h_{41k})^2\right] + (1 - \eta^2)\,\mathrm{Ric}(Je_1, e_2).$$

The backward heat kernel ρ_{y_0,t_0} satisfies the following parabolic equation along the mean curvature flow. Notice that ∇ and Δ are the covariant derivative and the Laplace operator on Σ_t respectively.

$$\frac{d}{dt}\rho_{y_0,t_0} = -\Delta\rho_{y_0,t_0} - \rho_{y_0,t_0}\left(\frac{|F^\perp|^2}{4(t_0-t)^2} + \frac{F^\perp \cdot \overline{H}}{t_0 - t} + \frac{F^\perp \cdot E}{2(t_0-t)}\right) \tag{5.3}$$

where $F^\perp$ is the component of $F \in T\mathbb{R}^N$ in $T\mathbb{R}^N/T\Sigma_t$. This equation for mean curvature flow in a Euclidean space is essentially derived by Huisken [3] and in a general ambient manifold by White [11]. It is derived in the next paragraph for completeness. Recall that

$$\frac{d}{dt}F(x,t) = H = \overline{H} + E$$

where $H \in TM/T\Sigma$ is the mean curvature vector of Σ_t in M and $\overline{H} \in T\mathbb{R}^N/T\Sigma$ is the mean curvature vector of Σ_t in $\mathbb{R}^N$.

We may assume y_0 is the origin and then

$$\rho_{y_0,t_0}(F(x,t),t) = \frac{1}{(4\pi(t_0-t))^{\frac{n}{2}}}\exp\left(\frac{-|F(x,t)|^2}{4(t_0-t)}\right).$$

Abbreviate $\rho_{y_0,t_0}(F(x,t),t)$ by ρ, it is not hard to see

$$\frac{d}{dt}\rho = \rho\left[\frac{n}{2(t_0-t)} - \frac{|F(x,t)|^2}{4(t_0-t)^2} - \frac{F \cdot H}{2(t_0-t)}\right]. \tag{5.4}$$

We shall compute $\sum_i(\overline{\nabla}_{e_i}\overline{\nabla}\rho)\cdot e_i$ in two different ways, where $\overline{\nabla}$ denotes the covariant derivative in $\mathbb{R}^N$ and $\{e_i\}$ is an orthonormal basis for $T\Sigma$.

$$\sum_i(\overline{\nabla}_{e_i}\overline{\nabla}\rho)\cdot e_i = \sum_i \overline{\nabla}_{e_i}(\nabla\rho + (\overline{\nabla}\rho)^{T\mathbb{R}^N/T\Sigma})\cdot e_i = \Delta\rho - \overline{\nabla}\rho\cdot\overline{H}.$$

$\overline{\nabla}\rho = -\frac{\rho}{2(t_0-t)}F$, thus

$$\sum_i (\overline{\nabla}_{e_i}\overline{\nabla}\rho)\cdot e_i = \Delta\rho + \frac{1}{2(t_0-t)}F\cdot\overline{H}.$$

On the other hand,

$$\begin{aligned}\sum_i (\overline{\nabla}_{e_i}\overline{\nabla}\rho)\cdot e_i &= \sum_i \overline{\nabla}_{e_i}\left(-\frac{\rho}{2(t_0-t)}F\right)\cdot e_i\\ &= -\frac{1}{2(t_0-t)}\left(\nabla\rho\cdot F + \rho\sum_i \overline{\nabla}_{e_i}F\cdot e_i\right)\\ &= -\frac{1}{2(t_0-t)}\left(-\frac{\rho}{2(t_0-t)}F^{T\Sigma}\cdot F + n\rho\right)\\ &= \rho\left(\frac{1}{4(t_0-t)^2}|F^{T\Sigma}|^2 - \frac{n}{2(t_0-t)}\right).\end{aligned}$$

Compare these two equalities, we get

$$\Delta\rho = \rho\left[-\frac{1}{2(t_0-t)}F\cdot\overline{H} + \frac{1}{4(t_0-t)^2}|F^{T\Sigma}|^2 - \frac{n}{2(t_0-t)}\right]. \tag{5.5}$$

Now add Equations (5.4) and (5.5), we get

$$\begin{aligned}\frac{d}{dt}\rho + \Delta\rho &= \frac{\rho}{4(t_0-t)^2}(|F^{T\Sigma}|^2 - |F|^2) - \frac{\rho}{2(t_0-t)}(F\cdot\overline{H} + F\cdot H)\\ &= -\frac{\rho}{4(t_0-t)^2}|F^{\perp}|^2 - \frac{\rho}{2(t_0-t)}(F^{\perp}\cdot\overline{H} + F^{\perp}\cdot H)\end{aligned}$$

where $F^{\perp} = (F)^{T\mathbb{R}^N/T\Sigma}$. Recall that $H = \overline{H} + E$ and we get Equation (5.3).

The minus sign in front of the Laplacian in Equation (5.3) indicates the fact that ρ satisfies the backward heat equation. The following inequality is particularly useful when deal with backward heat kernels:

$$g(-\Delta\rho) + (\Delta g)\rho = -\mathrm{div}(\nabla\rho\, g) + \mathrm{div}(\rho\nabla g). \tag{5.6}$$

The volume form $d\mu_t$ of Σ_t satisfies the equation

$$\frac{d}{dt}d\mu_t = -|H|^2 d\mu_t = -\overline{H}\cdot(\overline{H}+E)d\mu_t.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\int \psi(1-\eta)\rho_{y_0,t_0}\,d\mu_t = &\int \left[\frac{d}{dt}\psi(1-\eta)\right]\rho_{y_0,t_0}\,d\mu_t\\ &+\int \psi(1-\eta)\left[\frac{d}{dt}\rho_{y_0,t_0}\right]\,d\mu_t\\ &-\int \psi(1-\eta)\rho_{y_0,t_0}\,\overline{H}\cdot(\overline{H}+E)d\mu_t.\end{aligned}$$

Plug the Equation (5.3) for $\frac{d}{dt}\rho_{y_0,t_0}$, use the identity (5.6) with $g = \psi(1-\eta)$, and complete square we get

$$\begin{aligned}&\frac{d}{dt}\int \psi(1-\eta)\rho_{y_0,t_0}\,d\mu_t\\ &\quad=\int \left[\frac{d}{dt}(\psi(1-\eta))-\Delta(\psi(1-\eta))\right]\rho_{y_0,t_0}\,d\mu_t\\ &\quad\quad-\int \psi(1-\eta)\rho_{y_0,t_0}\left[\left|\overline{H}+\frac{1}{2(t_0-t)}F^{\perp}\right|^2\right.\\ &\quad\quad\quad\quad\left.+\left(\overline{H}+\frac{1}{2(t_0-t)}F^{\perp}\right)\cdot E\right]d\mu_t.\end{aligned} \tag{5.7}$$

Now

$$\begin{aligned}&\frac{d}{dt}(\psi(1-\eta))-\Delta(\psi(1-\eta))\\ &\quad=\psi\left(-\frac{d}{dt}\eta+\Delta\eta\right)+(\overline{\nabla}\psi\cdot H)(1-\eta)+2\nabla\psi\cdot\nabla\eta-\Delta\psi(1-\eta)\end{aligned}$$

where we use $\frac{d}{dt}\psi = \overline{\nabla}\psi\cdot H$.

Integration by parts gives

$$\begin{aligned}&\int[2\nabla\psi\cdot\nabla\eta-\Delta\psi(1-\eta)]\rho_{y_0,t_0}d\mu_t\\ &\quad\quad=\int[\nabla\psi\cdot\nabla\eta\,\rho_{y_0,t_0}+\nabla\psi\cdot\nabla\rho_{y_0,t_0}\,(1-\eta)]d\mu_t.\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{d}{dt}\int \psi(1-\eta)\rho_{y_0,t_0}\,d\mu_t\\
&\quad= -\int \psi\eta\rho_{y_0,t_0}[(h_{31k}-h_{42k})^2+(h_{32k}+h_{41k})^2]\,d\mu_t\\
&\qquad -\int \psi(1-\eta)\rho_{y_0,t_0}\left|\overline{H}+\frac{1}{2(t_0-t)}F^{\perp}+\frac{E}{2}\right|^2 d\mu_t\\
&\qquad +\int \psi(1-\eta)\rho_{y_0,t_0}\frac{|E|^2}{4}\,d\mu_t\\
&\qquad +\int [(\overline{\nabla}\psi\cdot H)(1-\eta)\rho_{y_0,t_0}+\nabla\psi\cdot\nabla\eta\,\rho_{y_0,t_0}\\
&\qquad\qquad +\nabla\psi\cdot\nabla\rho_{y_0,t_0}\,(1-\eta)]d\mu_t.
\end{aligned}$$

Since $|E|$ and $\int \rho_{y_0,t_0}d\mu_t$ are both bounded,

$$\begin{aligned}
&\frac{d}{dt}\int \psi(1-\eta)\rho_{y_0,t_0}\,d\mu_t\\
&\quad\le C-\int \psi\eta\rho_{y_0,t_0}[(h_{31k}-h_{42k})^2+(h_{32k}+h_{41k})^2]\,d\mu_t\\
&\qquad +\int [(\overline{\nabla}\psi\cdot H)(1-\eta)\rho_{y_0,t_0}+\nabla\psi\cdot\nabla\eta\,\rho_{y_0,t_0}\\
&\qquad\qquad +\nabla\psi\cdot\nabla\rho_{y_0,t_0}\,(1-\eta)]d\mu_t.
\end{aligned}$$

The last term is also bounded by the following computation:

$$\int \nabla\psi\cdot\nabla\rho_{y_0,t_0}\,(1-\eta)\,d\mu_t \le C\int_{B\setminus B_{\frac{1}{2}r}(y_0)} |\nabla\rho_{y_0,t_0}|d\mu_t.$$

Since $\nabla\rho_{y_0,t_0}=-\rho_{y_0,t_0}\frac{\nabla|F-y_0|^2}{4(t_0-t)}$ and

$$|\nabla|F-y_0|^2|\le|\overline{\nabla}|F-y_0|^2||\le 2|F-y_0|,$$

we have

$$\begin{aligned}
&\int \nabla\psi\cdot\nabla\rho_{y_0,t_0}\,(1-\eta)\,d\mu_t\\
&\qquad\qquad \le C\int_{B\setminus B_{\frac{1}{2}r}(y_0)}\frac{1}{(t_0-t)^{\frac{n}{2}+1}}\exp\left(\frac{-\frac{1}{4}r^2}{4(t_0-t)}\right)d\mu_t.
\end{aligned}$$

The last expression approaches zero as $t\to t_0$.

Therefore

$$\begin{aligned}
&\frac{d}{dt}\int \psi(1-\eta)\rho_{y_0,t_0}\,d\mu_t \\
&\quad \le C - \int \psi\eta\rho_{y_0,t_0}[(h_{31k}-h_{42k})^2+(h_{32k}+h_{41k})^2]\,d\mu_t \\
&\qquad + \int(\overline{\nabla}\psi\cdot H)(1-\eta)\rho_{y_0,t_0}d\mu_t + \int \nabla\psi\cdot\nabla\eta\,\rho_{y_0,t_0}d\mu_t.
\end{aligned}$$

The term $\int \nabla\psi\cdot\nabla\eta\,\rho_{y_0,t_0}\,d\mu_t$ can be written in the following

$$\begin{aligned}
&\int (\frac{\nabla\psi}{\sqrt{\psi}}\sqrt{\rho_{y_0,t_0}})\cdot(\nabla\eta\,\sqrt{\psi}\,\sqrt{\rho_{y_0,t_0}})\,d\mu_t \\
&\quad \le \frac{1}{4\epsilon^2}\int \frac{|\overline{\nabla}\psi|^2}{\psi}\rho_{y_0,t_0}\,d\mu_t + \epsilon^2\int |\nabla\eta|^2\psi\rho_{y_0,t_0}d\mu_t
\end{aligned}$$

where we use $|\nabla\psi|^2 \le |\overline{\nabla}\psi|^2$.

In a normal coordinate system, we compute $\nabla\eta$, again use the basis in Equation (4.1).

$$\begin{aligned}
\partial_k\eta =&\partial_k\left(\frac{\omega(\partial_1,\partial_2)}{\sqrt{\det g}}\right) \\
=&\partial_k(\omega(\partial_1,\partial_2)) \\
=&\omega(A(\partial_k,\partial_1),\partial_2)+\omega(\partial_1,A(\partial_k,\partial_2)) \\
=&h_{\alpha 1k}\omega_{\alpha 2}+h_{\alpha 2k}\omega_{1\alpha} \\
=&\sqrt{1-\eta^2}(h_{41k}+h_{32k}).
\end{aligned} \tag{5.8}$$

Therefore $|\nabla\eta|^2 \le (1-\eta^2)(h_{41k}+h_{32k})^2$ and thus

$$\begin{aligned}
&\int \nabla\psi\cdot\nabla\eta\,\rho_{y_0,t_0}\,d\mu_t \\
&\quad \le \frac{1}{4\epsilon^2}\int \frac{|\overline{\nabla}\psi|^2}{\psi}\rho_{y_0,t_0}\,d\mu_t + \epsilon^2\int(1-\eta^2)(h_{41k}+h_{32k})^2\psi\rho_{y_0,t_0}d\mu_t.
\end{aligned}$$

Likewise since $|H|^2 \le 2[(h_{31k}-h_{42k})^2+(h_{32k}+h_{41k})^2]$, we have

$$\begin{aligned}
&\int(\overline{\nabla}\psi\cdot H)(1-\eta)\rho_{y_0,t_0}d\mu_t \\
&\quad \le \frac{1}{4\epsilon^2}\int \frac{|\overline{\nabla}\psi|^2}{\psi}\rho_{y_0,t_0}\,d\mu_t + 2\epsilon^2\int(1-\eta)^2[(h_{31k}-h_{42k})^2 \\
&\qquad + (h_{41k}+h_{32k})^2]\psi\rho_{y_0,t_0}d\mu_t.
\end{aligned}$$

Since ψ is of compact support, by Lemma 6.6 in [5], $\frac{|\overline{\nabla}\psi|^2}{\psi} \leq 2\max|\overline{\nabla\nabla}\psi|$ is bounded.

Since $\eta > \delta$, we can choose ϵ small enough so that

$$\begin{aligned} &\frac{d}{dt}\int \psi(1-\eta)\rho_{y_0,t_0}\, d\mu_t \\ &\qquad \leq C - C_\delta \int \psi\rho_{y_0,t_0}[(h_{31k}-h_{42k})^2 + (h_{32k}+h_{41k})^2]\, d\mu_t \end{aligned} \tag{5.9}$$

where C_δ is a constant that depends on δ.

From this we see that $\lim_{t\to t_0}\int \psi(1-\eta)\rho_{y_0,t_0} d\mu_t$ exists.

For $\lambda > 1$, let's study the flow $\mathcal{S}^\lambda \subset \mathbb{R}^N \times [-\lambda^2 t_0, 0)$. Let $\rho^\lambda_{0,0,}(y,s)$ be the backward heat kernel at $(0,0)$ and $\psi^\lambda(F^\lambda_s(x)) = \psi(F_t(x))$. Recall that $t = t_0 + \frac{s}{\lambda^2}$. Thus

$$\begin{aligned} &\frac{d}{ds}\int \psi^\lambda(1-\eta^\lambda)\rho^\lambda_{0,0}\, d\mu^\lambda_s \\ &\quad = \frac{1}{\lambda^2}\frac{d}{dt}\int \psi(1-\eta)\rho_{y_0,t_0}\, d\mu_t \\ &\quad \leq \frac{C}{\lambda^2} - \frac{C_\delta}{\lambda^2}\int \psi\rho_{y_0,t_0}[(h_{31k}-h_{42k})^2 + (h_{32k}+h_{41k})^2]\, d\mu_t. \end{aligned}$$

We notice that η is a scaling invariant quantity therefore $\eta^\lambda = \eta$. It is not hard to check that

$$\begin{aligned} &\frac{1}{\lambda^2}\int \psi\rho_{y_0,t_0}[(h_{31k}-h_{42k})^2 + (h_{32k}+h_{41k})^2]\, d\mu_t \\ &\qquad = \int \psi^\lambda\rho^\lambda_{0,0}[(h^\lambda_{31k}-h^\lambda_{42k})^2 + (h^\lambda_{32k}+h^\lambda_{41k})^2]\, d\mu^\lambda_s. \end{aligned}$$

This is because $\rho_{y_0,t_0} d\mu_t$ is invariant under the parabolic scaling and the norm of second fundamental form scales like the inverse of the distance.

Therefore

$$\begin{aligned} &\frac{d}{ds}\int \psi^\lambda(1-\eta^\lambda)\rho^\lambda_{0,0}\, d\mu^\lambda_s \\ &\quad \leq \frac{C}{\lambda^2} - C_\delta\int \psi^\lambda\rho^\lambda_{0,0}[(h^\lambda_{31k}-h^\lambda_{42k})^2 + (h^\lambda_{32k}+h^\lambda_{41k})^2]\, d\mu^\lambda_s. \end{aligned}$$

Compare with Equation (5.9) and we see this reflects the correct scaling for the parabolic blow-up.

Take any $\tau > 0$ and integrate from $-1-\tau$ to -1.

$$
\begin{aligned}
(5.10)\quad & C_\delta \int_{-1-\tau}^{-1} \int \psi^\lambda \rho_{0,0}^\lambda \left[(h_{31k}^\lambda - h_{42k}^\lambda)^2 + (h_{32k}^\lambda + h_{41k}^\lambda)^2 \right] d\mu_s^\lambda ds \\
& \leq \int \psi^\lambda (1-\eta^\lambda) \rho_{0,0}^\lambda \, d\mu_{-1}^\lambda - \int \psi^\lambda (1-\eta^\lambda) \rho_{0,0}^\lambda \, d\mu_{-1-\tau}^\lambda + \frac{C}{\lambda^2}.
\end{aligned}
$$

Notice that

$$\int \psi^\lambda (1-\eta^\lambda) \rho_{0,0}^\lambda d\mu_s^\lambda = \int \psi (1-\eta) \rho_{y_0,t_0} d\mu_{t_0 + \frac{s}{\lambda^2}}.$$

This equality means the quantity $\int \psi(1-\eta)\rho_{y_0,t_0} d\mu_t$ is invariant under parabolic scaling. This fact is extremely important in applying the Monotonicity formula. Recall the natural Monotonicity formula for the volume

$$\frac{d}{dt} \int d\mu_t = - \int |H|^2 d\mu_t.$$

But $\int d\mu_s^\lambda = \lambda^2 \int d\mu_t$ is not scaling invariant. This deteriorates the usefulness of the formula in the blow-up analysis.

Now the right hand side in Equation (5.10) tends to zero as $\lambda \to \infty$. For any sequence $\lambda_i \to \infty$, we can choose $s_i \to -1$ such that

$$\int \psi^{\lambda_i} \rho_{0,0}^{\lambda_i} \left[(h_{31k}^{\lambda_i} - h_{42k}^{\lambda_i})^2 + (h_{32k}^{\lambda_i} + h_{41k}^{\lambda_i})^2 \right] d\mu_{s_i}^{\lambda_i} \to 0$$

as $i \to \infty$.

It is not hard to compute that

$$|A|^2(\mathcal{S}_s^{\lambda_i}) = \frac{1}{\lambda_i^2} |A|^2 \left(\mathcal{S}_{t_0 + \frac{s}{\lambda_i^2}} \right) = \left(\frac{-1}{s} \right) (t_0 - t_i) |A|^2 (\mathcal{S}_{t_i}).$$

The assumption implies each $\Sigma_s^{\lambda_i}$ has uniformly bounded second fundamental form. By the same method used in [3], any higher covariant derivatives of the second fundamental form of $\mathcal{S}_s^{\lambda_i}$ is bounded. Therefore the convergence $\mathcal{S}_{s_i}^{\lambda_i} \to \mathcal{S}_{-1}^\infty$ is smooth.

We may assume each $\mathcal{S}_t$ is connected by taking connected components. Therefore we have $(h_{31k} - h_{42k})^2 + (h_{32k} + h_{41k})^2 = 0$ for $\mathcal{S}_{-1}^\infty$. This implies $\nabla \eta = 0$ and $H = 0$. Applying the same argument to the monotonicity formula for $\int \psi \rho_{y_0,t_0} d\mu_t$ gives $H + \frac{1}{2} F^\perp = 0$ for $\mathcal{S}_{-1}^\infty$. To sum up, we get $F^{\perp} = 0$ and $\nabla\eta = 0$ for $\mathcal{S}^{\infty}_{-1}$. The first condition implies $\mathcal{S}^{\infty}_{-1}$ is a plane with multiplicity one. On the other hand,

$$\begin{aligned}\lim_{t_i \to t_0} \int \rho_{y_0,t_0} d\mu_{t_i} &= \lim_{i\to\infty} \int \frac{1}{4\pi(-s_i)} \exp\left(\frac{-|F^{\lambda_i}_{s_i}|^2}{4(-s_i)}\right) d\mu^{\lambda_i}_{s_i} \\ &= \int \frac{1}{4\pi} \exp\left(\frac{-|F^{\infty}_{-1}|^2}{4}\right) d\mu^{\infty}_{-1}\end{aligned}$$

where $\lambda_i = \sqrt{\frac{-s_i}{t_0 - t_i}}$.

The last Gaussian integral for a plane can be calculated directly and is equal to 1. By White's theorem, (y_0, t_0) is a regular point. q.e.d.

We recall the following definition of type I singularities for the mean curvature flow.

Definition 5.2. A singularity at t_0 is called type I if there exists a C such that $|A|^2 \leq \frac{C}{t_0 - t}$.

Recall a Kähler manifold M is called Kähler-Einstein if $\mathrm{Ric} = cg$ for some constant c. In this case, the scalar curvature s of M is a constant and $c = \frac{s}{4}$. A immersion $F : \Sigma \to M$ is symplectic if $\eta > 0$.

Theorem A. *Let M be a four-dimensional Kähler-Einstein manifold, then a symplectic surface remains symplectic along the mean curvature flow and the flow does not develop any type* I *singularities.*

Proof. Since $\mathrm{Ric}(J\cdot, \cdot) = \omega(\cdot, \cdot)$ by Proposition 4.1, the equation of $\eta = *\omega$ now becomes

$$\frac{d}{dt}\eta = \Delta\eta + \eta\left[(h_{31k} - h_{42k})^2 + (h_{32k} + h_{41k})^2 + c(1 - \eta^2)\right]. \tag{5.11}$$

The first assertion follows from maximum principle for parabolic equations. Actually, when $c \geq 0$, i.e., the nonnegative scalar curvature case, the function $\min_\Sigma \eta_t$ is a non-decreasing function of t. In any case, by comparison theorem for parabolic equations, η has a positive lower bound at any finite time and Proposition 5.2 is applicable. q.e.d.

Remark 5.1. The same argument can be used to prove there is no type I singularity for the mean curvature flow of an almost calibrated Lagrangian submanifolds in a Calabi-Yau manifold M. Here the almost calibrated condition is $*\Omega > 0$ where Ω is the real part of the canonical form on M. In fact, $*\Omega$ satisfies

$$\frac{d}{dt} * \Omega = \Delta * \Omega + |H|^2 * \Omega.$$

A smooth blow-up limit satisfies $H + \frac{1}{2}F^{\perp} = 0$ and $H = 0$ and is thus a linear subspace.

6. Long time existence and convergence

In this section, we study the problem of long time existence. The main result is the following.

Proposition 6.1. *Let M be an oriented four-dimensional compact manifold. Let ω' and ω'' be two parallel calibrating form such that ω' is self-dual and ω'' is anti-self-dual. Let $(\Sigma_0, d\mu) \hookrightarrow M$ be an compact surface with orientation $d\mu$. Let $F : \Sigma \times [0, t_0) \to M$ be the mean curvature flow of Σ_0 such that there exist a $\delta > 0$ with $*\omega' > \delta$ and $*\omega'' > \delta$ on $F_t(\Sigma)$ for $0 \le t < t_0$. Then F can be extended smoothly to $\Sigma \times [0, \bar{t}_0)$ for some $\bar{t}_0 > t_0$.*

Proof. The assumption implies $(\omega')^2$ and $(\omega'')^2$ determine different orientations on M. Choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for TM with $\{e_1, e_2\}$ a basis for $T\Sigma$ such that $(\omega')^2(e_1, e_2, e_3, e_4) > 0$ and $d\mu(e_1, e_2) > 0$.

Both ω' and ω'' are parallel calibrating forms and Proposition 4.1 is applicable. Therefore,

$$\begin{aligned}\frac{d}{dt}\eta' = \Delta\eta' &+ \eta'[(h_{31k} - h_{42k})^2 + (h_{32k} + h_{41k})^2] \\ &+ (1 - (\eta')^2)\operatorname{Ric}(J_1(e_1), e_2).\end{aligned}$$

On the other hand, by switching e_3 and e_4,

$$\begin{aligned}\frac{d}{dt}\eta'' = \Delta\eta'' &+ \eta''[(h_{41k} - h_{32k})^2 + (h_{42k} + h_{31k})^2] \\ &+ (1 - (\eta'')^2)\operatorname{Ric}(J_2(e_1), e_2).\end{aligned}$$

Adding these two equations and denote $\eta' + \eta''$ by μ, we get

$$\begin{aligned}\frac{d}{dt}\mu =&\Delta\mu + \mu|A|^2 + 2(\eta' - \eta'')h_{32k}h_{41k} - 2(\eta' - \eta'')h_{31k}h_{42k} \\ &+ (1 - (\eta')^2)\operatorname{Ric}(J_1(e_1), e_2) + (1 - (\eta'')^2)\operatorname{Ric}(J_2(e_1), e_2).\end{aligned}$$

Write $\mu = 2(\min\{\eta', \eta''\}) + |\eta' - \eta''|$, then $\mu \ge 2\delta + |\eta' - \eta''|$. After completing square, μ satisfies the following inequality:

$$\frac{d}{dt}\mu \ge \Delta\mu + 2\delta|A|^2 - C$$

where $-C$ is the lower bound of the Ricci curvature of M, $\mathrm{Ric} \geq -Cg$.

As before, we can isometrically embed M into $\mathbb{R}^N$. To detect a possible singularity at a point (y_0, t_0), where $y_0 \in M \hookrightarrow \mathbb{R}^N$ and $t_0 < \infty$, take a ball B of radius r about $y_0 \in \mathbb{R}^N$ and ψ a cut-off function as in the proof of Proposition 5.2. A similar argument yields the following inequality:

$$\frac{d}{dt}\int \psi(2-\mu)\rho_{y_0,t_0}\, d\mu_t \leq C - C_\delta \int \psi\rho_{y_0,t_0}|A|^2\, d\mu_t$$

where C_δ is a constant depend on δ.

Therefore $\lim_{t\to t_0}\int \psi\rho_{y_0,t_0}(2-\mu)d\mu_t$ exists. Let $\mathcal{S}^{\lambda_i}$ be a blow-up sequence at (y_0, t_0) that converges to $\mathcal{S}^\infty$. As in the previous section we can show for a fixed $\tau > 0$,

$$\int_{-1-\tau}^{-1}\int \psi^{\lambda_j}\rho_{0,0}^{\lambda_j}|A|^2 d\mu_s^{\lambda_j}\, ds \leq C(j)$$

where $C(j) \to 0$ as $\lambda_j \to \infty$.

Choose $\tau_j \to 0$ such that $\frac{C(j)}{\tau_j} \to 0$ and $s_j \in [-1-\tau_j, -1]$ so that

$$\int \psi^{\lambda_j}\rho_{0,0}^{\lambda_j}|A|^2 d\mu_{s_j}^{\lambda_j} \leq \frac{C(j)}{\tau_j}.$$

We investigate this inequality more carefully. Notice that ψ^{λ_j} is supported in $B_{\lambda_j r}(0) \subset \mathbb{R}^N$ and $\psi^{\lambda_j} \equiv 1$ in $B_{\frac{\lambda_j r}{2}}(0)$. Also

$$\rho_{0,0}^{\lambda_j}(F_{s_j}^{\lambda_j}) = \frac{1}{4\pi(-s_j)}\exp\left(\frac{-|F_{s_j}^{\lambda_j}|^2}{4(-s_j)}\right).$$

If we consider for any $R > 0$, the ball of radius R, $B_R(0) \subset \mathbb{R}^N$, when j is large enough, we may assume $\frac{\lambda_j r}{2} > R$ and $-1 < s_j < -\frac{1}{2}$, then

$$\int \psi^{\lambda_j}\rho_{0,0}^{\lambda_j}|A|^2 d\mu_{s_j}^{\lambda_j} \geq \frac{1}{2\pi}\exp\left(\frac{-R^2}{2}\right)\int_{\Sigma_{s_j}^{\lambda_j}\cap B_R(0)}|A|^2 d\mu_{s_j}^{\lambda_j}.$$

This implies for any compact set $K \subset \mathbb{R}^N$,

$$\int_{\Sigma_{s_j}^{\lambda_j}\cap K}|A|^2 d\mu_{s_j}^{\lambda_j} \to 0 \text{ as } j \to \infty.$$

Now we claim this together with the fact that μ has a positive lower bound imply $\lim_{j\to\infty}\int \rho_{y_0,t_0} d\mu_{t_0+\frac{s_j}{\lambda_j^2}} = \lim_{j\to\infty}\int \rho_{0,0} d\mu_{s_j}^{\lambda_j} \le 1$. We may assume the origin in $\mathbb{R}^N$ is a limit point of $\Sigma_{s_j}^{\lambda_j}$, otherwise the limit is zero.

We notice that $\omega' + \omega''$ is a parallel two form with $\lambda_1 = 2$ and $\lambda_2 = 0$ from the last paragraph in §3. Therefore the holonomy group of M splits into $SO(2)\times SO(2)$ and M is locally a Riemannian product. For simplicity, we shall assume M is a product $\Sigma_1 \times \Sigma_2$ such that $\frac{1}{2}(\omega' + \omega'')$ is the volume form of Σ_1. In fact, we can choose local coordinates (x^1, y^1) on Σ_1 and (x^2, y^2) on Σ_2 so that $\omega' = dx^1 \wedge dy^1 + dx^2 \wedge dy^2$, $\omega'' = dx^1 \wedge dy^1 - dx^2 \wedge dy^2$ and $\mu = 2(dx^1 \wedge dy^1)$.

Let $\pi_1 : \Sigma_1 \times \Sigma_2 \mapsto \Sigma_1$ be the projection. $\frac{\mu}{2}$ is in fact the Jacobian of the projection π_1 when restricted to Σ_t and the restriction $\pi_1|_{\Sigma_t}$ is a covering map. Now take any neighborhood Ω of $\pi_1(y_0) \in \Sigma_1$ and consider $\pi_1^{-1}(\Omega)\cap\Sigma_t$. Take any component and denote it by $\mathcal{S}_t$. $\mathcal{S}$ is an unparametrized flow. Each $\mathcal{S}_t$ can be written as the graph of a map $u_t : \Omega \mapsto \Sigma_2$ with uniformly bounded $|du_t|$ since μ_t has a uniform lower bound. Since y_0 is a limit point of $\Sigma_{t_0+\frac{s_j}{\lambda_j^2}}$, by choosing Ω small enough, we may assume the graph of $u_j = u_{t_0+\frac{s_j}{\lambda_j^2}}$ lies in B.

Now we consider the parabolic blow up of the graph of u_j in $\mathbb{R}^N$ by λ_j. This is the graph of the map $\widetilde{u}_j$ from $\lambda_j\Omega$ to $\lambda_j\Sigma_2$. It corresponds to a part of $\Sigma_{s_j}^{\lambda_j}$. By the assumption that the origin is a limit point of $\Sigma_{s_j}^{\lambda_j}$ and $|d\widetilde{u}_j|$ is uniformly bounded, we may assume $\widetilde{u}_j \to \widetilde{u}_\infty$ in C^α on compact sets. $\widetilde{u}_\infty$ is an entire graph defined on $\mathbb{R}^2$.

On the other hand,

$$|A|_j \le |\nabla d\widetilde{u}_j| \le (\sqrt{1+|d\widetilde{u}_j|^2})^3 |A|_j$$

where $|A|_j$ is the norm of the second fundamental form of $\mathcal{S}_{s_j}^{\lambda_j}$ and $|\nabla d\widetilde{u}_j|$ is the norm of the covariant derivatives of $d\widetilde{u}_j$. Now we identify Ω with an open set in $\mathbb{R}^2$. Therefore for any $B_\rho \subset \mathbb{R}^2$, $\widetilde{u}_j$ satisfies

$$|D\widetilde{u}_j| \le C, \quad \int_{B_\rho} |D^2\widetilde{u}_j|^2 \to 0$$

where $D\widetilde{u}_j$ and $D^2\widetilde{u}_j$ are the usual derivatives with respect to coordinate variables on $\mathbb{R}^2$. Denote $v_j = \frac{\partial \widetilde{u}_j}{\partial x_k}$, then $|v_j| \le C$ and $\int_{B_\rho} |Dv_j|^2 \to 0$.

Let $c_j = \frac{1}{\mathrm{Vol}(B_\rho)} \int v_j$, then we can choose a convergent subsequence $c_j \to c$. By Poincare inequality,

$$\int |v_j - c_j|^2 \leq \lambda \int |Dv_j|^2 \to 0.$$

Therefore $\frac{\partial \widetilde{u}_j}{\partial x^k} \to c_k$ in L^2. Since we may assume $\widetilde{u}_j \to \widetilde{u}_\infty$ in $C^\alpha \cap W^{1,2}_{\mathrm{loc}}$, this implies $\mathcal{S}^{\lambda_j}_{s_j} \to \mathcal{S}^\infty_{-1}$ as Radon measures and $\mathcal{S}^\infty_{-1}$ is the graph of a linear function. Therefore

$$\lim_{j\to\infty} \int \rho_{0,0} d\mu^{\lambda_j}_{s_j} = \int \rho_{0,0} d\mu^\infty_{-1} = 1.$$

By White's theorem again, we have regularity at the point (y_0, t_0).

q.e.d.

Now we prove Theorem B.

Theorem B. *Let M be an oriented four-dimensional Einstein manifold with two parallel calibrating forms ω', ω'' such that ω' is self-dual and ω'' is anti-self-dual. If Σ is a compact oriented surface immersed in M such that $*\omega', *\omega'' > 0$ on Σ. Then the mean curvature flow of Σ exists smoothly for all time.*

Proof. $*\omega'$ and $*\omega''$ have positive lower bound for any finite time by Equation (5.11), therefore the assumption in Proposition 6.1 is satisfied.

q.e.d.

7. Convergence at infinity

In this section we study the convergence of the mean curvature flow at infinity. The key point is to show uniform boundedness of $|A|^2$ in space and time. We first compute the evolution of the second fundamental form. Let $\Sigma \to M^n$ be an isometric immersion. We choose an orthonormal basis $\{e_i\}$ for $T\Sigma$ and $\{e_\alpha\}$ for $N\Sigma$. Recall the convention for indexes are $A, B, C \cdots = 1 \cdots n$, $i, j, k \cdots$ for tangent indexes, and $\alpha, \beta, \gamma \cdots$ for normal indexes. Now denote the coefficient of the second fundamental form by $h_{\alpha ij} = \langle A(\partial_i, \partial_j), e_\alpha \rangle$. The covariant derivative of A is defined as

$$(\overline{\nabla}_{\partial_l} A)(\partial_i, \partial_j) = (\overline{\nabla}_{\partial_l} A(\partial_i, \partial_j))^N - A((\overline{\nabla}_{\partial_l} \partial_i)^T, \partial_j) - A(\partial_i, (\overline{\nabla}_{\partial_l} \partial_j)^T).$$

We denote

$$h_{\alpha ij,k} = \langle (\overline{\nabla}_{\partial_k} A)(\partial_i, \partial_j), e_\alpha \rangle$$
$$h_{\alpha ij,kl} = \langle (\overline{\nabla}_{\partial_l} \overline{\nabla}_{\partial_k} A)(\partial_i, \partial_j), e_\alpha \rangle.$$

Let $\Delta h_{\alpha ij} = g^{kl} h_{\alpha ij,kl}$ be the Laplacian of $h_{\alpha ij}$.

Proposition 7.1. *For a mean curvature flow $F : \Sigma \times [0, t_0) \to M$ of any dimension, the second fundamental form $h_{\alpha ij}$ satisfies the following equation.*

$$\begin{aligned} \frac{d}{dt} h_{\alpha ij} =& \Delta h_{\alpha ij} + (\overline{\nabla}_{\partial_k} K)_{\alpha ijk} + (\overline{\nabla}_{\partial_j} K)_{\alpha kik} \\ & - 2K_{lijk} h_{\alpha lk} + 2K_{\alpha\beta jk} h_{\beta ik} + 2K_{\alpha\beta ik} h_{\beta jk} \\ & - K_{lkik} h_{\alpha lj} - K_{lkjk} h_{\alpha li} + K_{\alpha k\beta k} h_{\beta ij} \\ & - h_{\alpha im}(h_{\gamma mj} h_\gamma - h_{\gamma mk} h_{\gamma jk}) \\ & - h_{\alpha mk}(h_{\gamma mj} h_{\gamma ik} - h_{\gamma mk} h_{\gamma ij}) \\ & - h_{\beta ik}(h_{\beta lj} h_{\alpha lk} - h_{\beta lk} h_{\alpha lj}) \\ & - h_{\alpha jk} h_{\beta ik} h_\beta + h_{\beta ij} \langle e_\beta, \overline{\nabla}_H e_\alpha \rangle \end{aligned} \tag{7.1}$$

where K_{ABCD} is the curvature tensor and $\overline{\nabla}$ is the covariant derivative of M.

In particular, $|A|^2$ satisfies the following equation along the mean curvature flow.

$$\begin{aligned} \frac{d}{dt}|A|^2 =& \Delta |A|^2 - 2|\nabla A|^2 + 2[(\overline{\nabla}_{\partial_k} K)_{\alpha ijk} + (\overline{\nabla}_{\partial_j} K)_{\alpha kik}] h_{\alpha ij} \\ & - 4K_{lijk} h_{\alpha lk} h_{\alpha ij} + 8K_{\alpha\beta jk} h_{\beta ik} h_{\alpha ij} \\ & - 4K_{lkik} h_{\alpha lj} h_{\alpha ij} + 2K_{\alpha k\beta k} h_{\beta ij} h_{\alpha ij} \\ & + 2 \sum_{\alpha,\gamma,i,m} (\sum_k h_{\alpha ik} h_{\gamma mk} - h_{\alpha mk} h_{\gamma ik})^2 \\ & + 2 \sum_{i,j,m,k} (\sum_\alpha h_{\alpha ij} h_{\alpha mk})^2. \end{aligned} \tag{7.2}$$

Proof. We first derive Equation (7.2) from Equation (7.1).

Since $|A|^2 = g^{ik} g^{jl} h_{\alpha ij} h_{\alpha kl}$, calculate using a normal coordinate system near a point p we have

$$\frac{d}{dt}|A|^2 = 2\left(\frac{d}{dt} g^{ik}\right) h_{\alpha ij} h_{\alpha kj} + 2\left(\frac{d}{dt} h_{\alpha ij}\right) h_{\alpha ij}.$$

Recall $\frac{d}{dt}g_{ik} = 2h_\beta h_{\beta ik}$ and plug in Equation (7.1) to get

$$\begin{aligned}\frac{d}{dt}|A|^2 =& 4h_\beta h_{\beta ik} h_{\alpha ij} h_{\alpha kj}\\ &+ 2h_{\alpha ij}[\Delta h_{\alpha ij} + (\overline{\nabla}_{\partial_k} K)_{\alpha ijk} + (\overline{\nabla}_{\partial_j} K)_{\alpha kik}\\ &- 2K_{lijk}h_{\alpha lk} + 2K_{\alpha\beta jk}h_{\beta ik} + 2K_{\alpha\beta ik}h_{\beta jk}\\ &- K_{lkik}h_{\alpha lj} - K_{lkjk}h_{\alpha li} + K_{\alpha k\beta k}h_{\beta ij}\\ &- h_{\alpha im}(h_{\gamma mj}h_\gamma - h_{\gamma mk}h_{\gamma jk})\\ &- h_{\alpha mk}(h_{\gamma mj}h_{\gamma ik} - h_{\gamma mk}h_{\gamma ij})\\ &- h_{\beta ik}(h_{\beta lj}h_{\alpha lk} - h_{\beta lk}h_{\alpha lj})\\ &- h_{\alpha jk}h_{\beta ik}h_\beta + h_{\beta ij}\langle e_\beta, \overline{\nabla}_H e_\alpha\rangle].\end{aligned}$$

The first term on the right hand side $4h_\beta h_{\beta ik} h_{\alpha ij} h_{\alpha kj}$ cancels with two later terms. They are so-called "metric" terms and vanish if we choose a orthonormal frame in our computation.

The last term on the right hand side $2h_{\alpha ij}h_{\beta ij}\langle e_\beta, \overline{\nabla}_H e_\alpha\rangle$ is zero by symmetry.

Now use $\Delta h_{\alpha ij}^2 = 2|\nabla A|^2 + 2h_{\alpha ij}\Delta h_{\alpha ij}$. Therefore we get

$$\begin{aligned}\frac{d}{dt}|A|^2 =& \Delta|A|^2 - 2|\nabla A|^2 + 2h_{\alpha ij}[(\overline{\nabla}_{\partial_k} K)_{\alpha ijk} + (\overline{\nabla}_{\partial_j} K)_{\alpha kik}\\ &- 2K_{lijk}h_{\alpha lk} + 2K_{\alpha\beta jk}h_{\beta ik} + 2K_{\alpha\beta ik}h_{\beta jk}\\ &- K_{lkik}h_{\alpha lj} - K_{lkjk}h_{\alpha li} + K_{\alpha k\beta k}h_{\beta ij}\\ &+ h_{\alpha im}h_{\gamma mk}h_{\gamma jk} - h_{\alpha mk}(h_{\gamma mj}h_{\gamma ik}\\ &- h_{\gamma mk}h_{\gamma ij}) - h_{\beta ik}(h_{\beta lj}h_{\alpha lk} - h_{\beta lk}h_{\alpha lj})].\end{aligned}$$

The fourth order terms can be calculated as the following:

$$\begin{aligned}&h_{\alpha ij}h_{\alpha im}h_{\gamma mk}h_{\gamma jk} - h_{\alpha ij}h_{\alpha mk}(h_{\gamma mj}h_{\gamma ik} - h_{\gamma mk}h_{\gamma ij})\\ &\qquad - h_{\alpha ij}h_{\beta ik}(h_{\beta lj}h_{\alpha lk} - h_{\beta lk}h_{\alpha lj})\\ &\quad = 2h_{\alpha ij}h_{\alpha im}h_{\gamma mk}h_{\gamma kj} - 2h_{\alpha ij}h_{\alpha mk}h_{\gamma mj}h_{\gamma ik} + h_{\alpha ij}h_{\alpha mk}h_{\gamma mk}h_{\gamma ij}.\end{aligned}$$

The first two terms can be completed to square:

$$\begin{aligned}&2h_{\alpha ij}h_{\alpha im}h_{\gamma mk}h_{\gamma kj} - 2h_{\alpha ij}h_{\alpha mk}h_{\gamma mj}h_{\gamma ik}\\ &\quad = 2h_{\alpha ij}h_{\alpha ik}h_{\gamma mk}h_{\gamma mj} - 2h_{\alpha ij}h_{\alpha mk}h_{\gamma mj}h_{\gamma ik}\\ &\quad = 2h_{\alpha ij}h_{\gamma mj}(h_{\alpha ik}h_{\gamma mk} - h_{\alpha mk}h_{\gamma ik})\\ &\quad = h_{\alpha ij}h_{\gamma mj}(h_{\alpha ik}h_{\gamma mk} - h_{\alpha mk}h_{\gamma ik})\\ &\qquad + h_{\alpha mj}h_{\gamma ij}(h_{\alpha mk}h_{\gamma ik} - h_{\alpha ik}h_{\gamma mk})\\ &\quad = \sum_{\alpha,\gamma,i,m}\left(\sum_k h_{\alpha ik}h_{\gamma mk} - h_{\alpha mk}h_{\gamma ik}\right)^2.\end{aligned} \tag{7.3}$$

Now we calculate Equation (7.1). First the Laplacian of $h_{\alpha ij}$ is the following.

$$
\begin{aligned}
\Delta h_{\alpha ij} =& h_{\alpha,ij} - (\nabla_{\partial_k} K)_{\alpha ijk} - (\nabla_{\partial_j} K)_{\alpha kik} \\
&+ 2K_{lijk}h_{\alpha lk} - 2K_{\alpha\beta jk}h_{\beta ik} - 2K_{\alpha\beta ik}h_{\beta jk} - K_{\alpha ij\beta}h_{\beta} \\
&+ K_{lkik}h_{\alpha lj} + K_{lkjk}h_{\alpha li} - K_{\alpha k\beta k}h_{\beta ij} \\
&+ h_{\alpha im}(h_{\gamma mj}h_{\gamma} - h_{\gamma mk}h_{\gamma jk}) \\
&+ h_{\alpha mk}(h_{\gamma mj}h_{\gamma ik} - h_{\gamma mk}h_{\gamma ij}) \\
&+ h_{\beta ik}(h_{\beta lj}h_{\alpha lk} - h_{\beta lk}h_{\alpha lj})
\end{aligned}
\tag{7.4}
$$

where $h_{\alpha,ij} = \langle \overline{\nabla}^N_{\partial_j}\overline{\nabla}^N_{\partial_i} H, e_\alpha\rangle$.

In the codimension one case, this equation reduces to

$$
\begin{aligned}
\Delta h_{ij} =& H_{,ij} - (\nabla_{\partial_k} K)_{Nijk} - (\nabla_{pj} K)_{Nkik} \\
&+ 2K_{lijk}h_{lk} - K_{Nijn+1}H \\
&+ K_{lkik}h_{lj} + K_{lkjk}h_{li} - K_{NkNk}h_{ij} \\
&+ h_{im}h_{mj}H - h^2_{mk}h_{ij}.
\end{aligned}
$$

This recovers Equation (1.20) in [6].

Equation (7.4) is computed using the Codazzi equation and the commutation formula.

$$
\begin{aligned}
h_{\alpha kj,i} &= h_{\alpha ki,j} + K_{\alpha kij} \\
h_{\alpha ij,kl} &= h_{\alpha ij,lk} - h_{\alpha im}R_{mjlk} - h_{\alpha mj}R_{milk} - h_{\beta ij}R_{\beta\alpha lk}
\end{aligned}
$$

where R_{mjlk} is the curvature of $T\Sigma$ and $R_{\alpha\beta lk}$ is the curvature of $N\Sigma$.

We start the computation with $h_{\alpha,ij} = h_{\alpha kk,ij}$.

$$
\begin{aligned}
h_{\alpha kk,ij} &= (h_{\alpha ki,k} + K_{\alpha kik})_j \\
&= h_{\alpha ki,kj} + K_{\alpha kik,j} \\
&= h_{\alpha ik,jk} - h_{\alpha im}R_{mkjk} - h_{\alpha mk}R_{mijk} \\
&\quad - h_{\beta ik}R_{\beta\alpha jk} + K_{\alpha kik,j} \\
&= (h_{\alpha ij,k} + K_{\alpha ijk})_k - h_{\alpha im}R_{mkjk} - h_{\alpha mk}R_{mijk} \\
&\quad - h_{\beta ik}R_{\beta\alpha jk} + K_{\alpha kik,j}.
\end{aligned}
$$

By the Gauss and Ricci equation, we have

$$
\begin{aligned}
R_{mkjk} &= K_{mkjk} + h_{\gamma mj}h_{\gamma kk} - h_{\gamma mk}h_{\gamma kj} \\
R_{mijk} &= K_{mijk} + h_{\gamma mj}h_{\gamma ik} - h_{\gamma mk}h_{\gamma ij} \\
R_{\beta\alpha jk} &= K_{\beta\alpha jk} + h_{\beta lj}h_{\alpha lk} - h_{\beta lk}h_{\alpha lj}.
\end{aligned}
$$

Therefore

$$\begin{aligned}\Delta h_{\alpha ij} =& h_{\alpha,ij} - K_{\alpha ijk,k} - K_{\alpha kik,j} + h_{\alpha im}K_{mkjk}\\ &+ h_{\alpha mk}K_{mijk} + h_{\beta ik}K_{\beta\alpha jk}\\ &+ h_{\alpha im}(h_{\gamma mj}h_{\gamma} - h_{\gamma mk}h_{\gamma jk})\\ &+ h_{\alpha mk}(h_{\gamma mj}h_{\gamma ik} - h_{\gamma mk}h_{\gamma ij})\\ &+ h_{\beta ik}(h_{\beta lj}h_{\alpha lk} - h_{\beta lk}h_{\alpha lj}).\end{aligned}$$

The covariant derivative term can be calculated as follows:

$$\begin{aligned}K_{\alpha ijk,k} =& (\overline{\nabla}_{\partial_k}K)_{\alpha ijk} - K_{lijk}h_{\alpha lk} + K_{\alpha\beta jk}h_{\beta ik}\\ &+ K_{\alpha i\beta k}h_{\beta jk} + K_{\alpha ij\beta}h_{\beta kk}\\ K_{\alpha kik,j} =& (\overline{\nabla}_{\partial_j}K)_{\alpha kik} - K_{lkik}h_{\alpha lj} + K_{\alpha\beta ik}h_{\beta kj}\\ &+ K_{\alpha k\beta k}h_{\beta ij} + K_{\alpha ki\beta}h_{\beta kj}.\end{aligned}$$

Note that $K_{\alpha ijk}$ is considered as a section of the bundle $N\Sigma\otimes T\Sigma\otimes T\Sigma\otimes T\Sigma$ in taking covariant derivatives . We collect all the embient curvature term and use the first Bianchi identity $K_{\alpha i\beta k} + K_{\alpha ki\beta} = -K_{\alpha\beta ki}$ to get Equation (7.4).

Next we calculate the equation for $h_{\alpha ij} = \langle\overline{\nabla}_{\partial_j}\partial_i, e_\alpha\rangle$.

$$\begin{aligned}\frac{d}{dt}h_{\alpha ij} &= \langle\overline{\nabla}_H\overline{\nabla}_{\partial_j}\partial_i, e_\alpha\rangle + \langle\overline{\nabla}_{\partial_j}\partial_i, \overline{\nabla}_H e_\alpha\rangle\\ &= \langle\overline{\nabla}_{\partial_j}\overline{\nabla}_H\partial_i, e_\alpha\rangle - \langle K(H,\partial_j)\partial_i, e_\alpha\rangle + \langle\overline{\nabla}_{\partial_j}\partial_i, \overline{\nabla}_H e_\alpha\rangle.\end{aligned}$$

By breaking $\overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_i}H$ into normal and tangent parts, we get

$$\begin{aligned}\langle\overline{\nabla}_{\partial_j}\overline{\nabla}_{\partial_i}H, e_\alpha\rangle &= \langle\overline{\nabla}_{\partial_j}\left[(\overline{\nabla}_{\partial_i}H)^T + (\overline{\nabla}_{\partial_i}H)^N\right], e_\alpha\rangle\\ &= \langle\overline{\nabla}^N_{\partial_j}\overline{\nabla}^N_{\partial_i}H, e_\alpha\rangle - \langle(\overline{\nabla}_{\partial_i}H)^T, \overline{\nabla}_{\partial_j}e_\alpha\rangle.\end{aligned}$$

Therefore,

$$\frac{d}{dt}h_{\alpha ij} = h_{\alpha,ij} - h_\beta K_{\beta ji\alpha} - \langle(\overline{\nabla}_{\partial_i}H)^T, \overline{\nabla}_{\partial_j}e_\alpha\rangle + \langle\overline{\nabla}_{\partial_j}\partial_i, \overline{\nabla}_H e_\alpha\rangle$$

where $h_{\alpha,ij} = \langle\overline{\nabla}^N_{\partial_j}\overline{\nabla}^N_{\partial_i}H, e_\alpha\rangle$.

The term $\langle(\overline{\nabla}_{\partial_i}H)^T, \overline{\nabla}_{\partial_j}e_\alpha\rangle$ is equal to $h_\beta h_{\beta ik}h_{\alpha jk}$. Also since we choose a normal coordinate in our computation, $(\overline{\nabla}_{\partial_i}\partial_j)^T = 0$ and $\langle\overline{\nabla}_{\partial_i}\partial_j, \overline{\nabla}_H e_\alpha\rangle = h_{\beta ij}\langle e_\beta, \overline{\nabla}_H e_\alpha\rangle$.

$$\frac{d}{dt}h_{\alpha ij} = h_{\alpha,ij} - h_\beta h_{\beta ik}h_{\alpha jk} - h_\beta K_{\beta ji\alpha} + h_{\beta ij}\langle e_\beta, \overline{\nabla}_H e_\alpha\rangle. \tag{7.5}$$

Combine Equation (7.4) and (7.5), we get the parabolic equation for $h_{\alpha ij}$. q.e.d.

The following proposition provides a uniform bound of the second fundamental form when $*\omega'$ and $*\omega''$ are both close to one.

Proposition 7.2. *Let M be a compact four-dimensional manifold with bounded geometry. Let ω' and ω'' be two parallel calibrating forms such that $\omega' \wedge \omega'$ and $\omega'' \wedge \omega''$ determine opposite orientation for M. Let Σ be an oriented immersed surface in M. There exists a constant $1 > \epsilon > 0$ such that if $*\omega' > 1 - \epsilon$ and $*\omega'' > 1 - \epsilon$ on $F_t(\Sigma)$ for $t \in [0, T]$, then the norm of the second fundamental form of $F_t(\Sigma)$ is uniformly bounded in $[0, T]$.*

Proof. The fourth order term in Equation (7.2) can be calculated explicitly in the four-dimensional case:

(7.6)

$$\begin{aligned}
&\sum_{\alpha,\gamma,i,m}\left(\sum_k h_{\alpha ik}h_{\gamma mk} - h_{\alpha mk}h_{\gamma ik}\right)^2\\
&= \sum_{i,m}\left(\sum_k h_{3ik}h_{4mk} - h_{3mk}h_{4ik}\right)^2 + \sum_{i,m}\left(\sum_k h_{4ik}h_{3mk} - h_{4mk}h_{3ik}\right)^2\\
&= 2\sum_{i,m}\left(\sum_k h_{3ik}h_{4mk} - h_{3mk}h_{4ik}\right)^2\\
&= 2\left[\left(\sum_k h_{31k}h_{42k} - h_{32k}h_{41k}\right)^2 + \left(\sum_k h_{41k}h_{32k} - h_{42k}h_{31k}\right)^2\right]\\
&= 4\left(\sum_k h_{31k}h_{42k} - h_{32k}h_{41k}\right)^2\\
&= 4\left[\frac{1}{2}(h_{31k}+h_{42k})^2 + \frac{1}{2}(h_{32k}-h_{41k})^2 - \frac{1}{2}|A|^2\right]^2\\
&= \left[(h_{31k}+h_{42k})^2 + (h_{32k}-h_{41k})^2 - |A|^2\right]^2.
\end{aligned}$$

By the Schwarz inequality,

$$\sum_{i,j,m,k}\left(\sum_\alpha h_{\alpha ij}h_{\alpha mk}\right)^2 \le \sum_{i,j,m,k}\left(\sum_\alpha h_{\alpha ij}^2\right)\left(\sum_\alpha h_{\alpha mk}^2\right) \le |A|^4.$$

Therefore

$$\frac{d}{dt}|A|^2 \le \Delta|A|^2 - 2|\nabla A|^2 + 4|A|^4 + K_1|A|^2 + K_2 \tag{7.7}$$

where K_1 and K_2 are constants that depend on the curvature tensor and covariant derivatives of the curvature tensor of M.

Again we consider $\mu = \eta' + \eta''$. Since $\eta' \ge 1 - \epsilon$ and $\eta'' \ge 1 - \epsilon$, we have $\mu \ge 2 - 2\epsilon$ and $|\eta' - \eta''| \le \epsilon \le \frac{\epsilon}{2-2\epsilon}\mu$.

$$\begin{aligned}\frac{d}{dt}\mu \ge& \Delta\mu + \mu|A|^2 + 2(\eta' - \eta'')h_{32k}h_{41k} - 2(\eta' - \eta'')h_{31k}h_{42k} - C\mu\\ \ge& \Delta\mu + \mu\left(|A|^2 - 2\frac{\epsilon}{2-2\epsilon}|h_{32k}h_{41k}| - 2\frac{\epsilon}{2-2\epsilon}|h_{31k}h_{42k}|\right) - C\mu\\ \ge& \Delta\mu + \mu\left[\frac{2-3\epsilon}{2-2\epsilon}|A|^2 + \frac{\epsilon}{2-2\epsilon}(|A|^2 - 2|h_{32k}h_{41k}| - 2|h_{31k}h_{42k}|)\right]\\ &- C\mu.\end{aligned}$$

The term $(|A|^2 - 2|h_{32k}h_{41k}| - 2|h_{31k}h_{42k}|)$ is a complete square and thus nonnegative.

Therefore

$$\frac{d}{dt}\mu \ge \Delta\mu + c_1(\epsilon)\mu|A|^2 - C\mu$$

where $c_1(\epsilon) = \frac{2-3\epsilon}{2-2\epsilon}$, notice that $c_1(\epsilon) \to 1$ as $\epsilon \to 0$.

Let $p > 1$ be an integer to be determined, we calculate the equation for μ^p.

$$\frac{d}{dt}\mu^p = p\mu^{p-1}\frac{d}{dt}\mu \ge p\mu^{p-1}(\Delta\mu + c_1(\epsilon)\mu|A|^2 - C\mu).$$

Use the identity $\Delta\mu^p = p(p-1)\mu^{p-2}|\nabla\mu|^2 + p\mu^{p-1}\Delta\mu$, the differential inequality for μ^p becomes

$$\frac{d}{dt}\mu^p \ge \Delta\mu^p - p(p-1)\mu^{p-2}|\nabla\mu|^2 + p\, c_1(\epsilon)\mu^p|A|^2 - Cp\mu^p.$$

Now we estimate the term $|\nabla\mu|^2$.

We calculate as in Equation (5.8):

$$\begin{aligned}|\nabla\eta'|^2 &\le 2(1 - (\eta')^2)(h_{41k}^2 + h_{32k}^2)\\ |\nabla\eta''|^2 &\le 2(1 - (\eta'')^2)(h_{31k}^2 + h_{42k}^2).\end{aligned}$$

Since $\eta' \geq 1-\epsilon$, $1-(\eta')^2 \leq 1-(1-\epsilon)^2 \leq 2\epsilon$. Therefore

$$|\nabla\mu|^2 \leq 2(|\nabla\eta'|^2 + |\nabla\eta''|^2) \leq 8\epsilon|A|^2 \leq \frac{8\epsilon}{(2-2\epsilon)^2}\mu^2|A|^2.$$

Denote $\frac{8\epsilon}{(2-2\epsilon)^2} = c_2(\epsilon)$.

Thus

$$\frac{d}{dt}\mu^p \geq \Delta\mu^p + p[c_1(\epsilon) - (p-1)c_2(\epsilon)]\mu^p|A|^2 - Cp\mu^p.$$

Plug in $f = |A|^2$ and $g = \mu^p$ in the identity

$$\frac{d}{dt}\left(\frac{f}{g}\right) = \Delta\left(\frac{f}{g}\right) + 2\frac{\nabla g}{g}\cdot\nabla\left(\frac{f}{g}\right) + \frac{1}{g^2}\left[\left(\frac{d}{dt}f - \Delta f\right)g - \left(\frac{d}{dt}g - \Delta g\right)f\right].$$

Therefore

$$\begin{aligned}\frac{d}{dt}\left(\frac{|A|^2}{\mu^p}\right) \leq \Delta\left(\frac{|A|^2}{\mu^p}\right) &+ 2\nabla\left(\frac{|A|^2}{\mu^p}\right)\cdot\frac{\nabla\mu^p}{\mu^p} \\ &+ \frac{1}{\mu^{2p}}\Big\{[-2|\nabla A|^2 + 4|A|^4 + K_1|A|^2 + K_2]\mu^p \\ &\qquad - [p(c_1(\epsilon) - (p-1)c_2(\epsilon))\mu^p|A|^2 - Cp\mu^p]|A|^2\Big\}.\end{aligned}$$

The last term is less than

$$[4 - p(c_1(\epsilon) - (p-1)c_2(\epsilon))]\frac{|A|^4}{\mu^p} + (K_1 + Cp)\frac{|A|^2}{\mu^p} + K_2\frac{1}{\mu^p}.$$

Recall that $c_1(\epsilon) \to 1$ and $c_2(\epsilon) \to 0$ as $\epsilon \to 0$.

Choose p large enough and then ϵ small enough so that

$$(2-2\epsilon)^p[4 - p(c_1(\epsilon) - (p-1)c_2(\epsilon))] \leq -C_1$$

for some $C_1 > 0$.

Then

$$\begin{aligned}[4 - p(c_1(\epsilon) - (p-1)c_2(\epsilon))]\frac{|A|^4}{\mu^p} &= [4 - p(c_1(\epsilon) - (p-1)c_2(\epsilon))]\mu^p\frac{|A|^4}{\mu^{2p}} \\ &\leq -C_1\frac{|A|^4}{\mu^{2p}}.\end{aligned}$$

Denote $f = \frac{|A|^2}{\mu^p}$, then f satisfies

$$\frac{d}{dt} f \leq \Delta f + V \cdot \nabla f - C_1 f^2 + C_2 f + C_3.$$

Now we apply the maximum principle for parabolic equations and conclude the $\frac{|A|^2}{\mu}$ is uniformly bounded, thus $|A|^2$ is also bounded.

q.e.d.

Now we prove Theorem C.

Theorem C. *Under the same assumption as in Theorem B. When M has nonnegative curvature, there exists a constant $\epsilon > 0$ such that if Σ is a compact oriented surface immersed in M with $*\omega', *\omega'' > 1-\epsilon$ on Σ, the mean curvature flow of Σ converges smoothly to a totally geodesic surface at infinity.*

Proof. In these cases, $*\omega'$ and $*\omega''$ are both non-decreasing. Proposition 7.2 is applicable and $|A|^2$ is uniformly bounded in space and time.

Integrating Equation (7.7) and we see

$$\frac{d}{dt} \int_{\Sigma_t} |A|^2 d\mu_t \leq C. \tag{7.8}$$

Recall in this case $\frac{d}{dt}\mu \geq \Delta\mu + c_1(\epsilon)\mu|A|^2$ and η has a positive lower bound, thus

$$\int_0^\infty \int_{\Sigma_t} |A|^2 d\mu_t dt \leq \infty. \tag{7.9}$$

Equation (7.8) and (7.9) together implies

$$\int_{\Sigma_t} |A|^2 d\mu_t \to 0.$$

By the small ϵ regularity theorem in [4], $\sup_{\Sigma_t} |A|^2 \to 0$ uniformly as $t \to \infty$.

Since the mean curvature flow is a gradient flow and the metrics are analytic, by the theorem of Simon [7], we get convergence at infinity.

q.e.d.

8. Applications

In the following we apply the previous theorem to the case when M is a product $S^2 \times S^2$. Denote their Kähler forms by ω_1 and ω_2 respectively.

Let $\omega' = \omega_1 + \omega_2$ and $\omega'' = \omega_1 - \omega_2$, then $(\omega')^2 = 2\omega_1 \wedge \omega_2$ and $(\omega'')^2 = -2\omega_1 \wedge \omega_2$ determine opposite orientations on M. Both ω' and ω'' are parallel calibrating form and they define integrable almost complex structures with opposite orientations.

Theorem D. *Let $M = (S^2, \omega_1) \times (S^2, \omega_2)$. If Σ is a compact oriented surface embedded in M such that $*\omega_1 \geq |*\omega_2|$ and the strict inequality holds at at least one point. Then the mean curvature flow of Σ exists for all time and converges smoothly to a $S^2 \times \{p\}$.*

Proof. We notice the statement is a little bit different from the one given in the introduction. The difference is resolved by the considering the maximum principle for $\eta' = *\omega'$ and $\eta'' = *\omega''$. It is not hard to see the assumption implies $*\omega_1 > |*\omega_2|$ holds everywhere at a later time. By the equation of η' and η'',

$$\frac{d}{dt}\eta' = \Delta\eta' + \eta'[(h_{31k} - h_{42k})^2 + (h_{32k} + h_{41k})^2] + \eta'(1 - (\eta')^2)$$
$$\frac{d}{dt}\eta'' = \Delta\eta'' + \eta''[(h_{41k} - h_{32k})^2 + (h_{42k} + h_{31k})^2] + \eta''(1 - (\eta'')^2)$$

we see that as $t \to \infty$, they both approach 1. By Theorem A, we have existence for all time. Also the assumption on Proposition 7.2 is satisfied and the second fundamental form is uniformly bounded in space and time. Since the mean curvature flow is a gradient flow and the metrics are analytic, we can apply Simon's theorem [7] to conclude convergence at infinity. The limiting submanifold has $*\omega_1 = 1$ identically and thus is of the form $S^2 \times p$. q.e.d.

Corollary D now follows from this since the condition $*\omega_1 > |*\omega_2|$ on the graph of a map f is equivalent to the Jacobian of f being less than one.

We conclude this section by the following two remarks:

Remark 8.1. When M is locally a product of two Riemann surfaces of nonpositive curvature, the method in [8] can be used to prove uniform convergence of the flow. The limit is totally geodesic and the corresponding map converges to one ranged in a lower dimensional submanifold. Notice that the convergence in Theorem C is a stronger smooth convergence under the closeness assumption. Such results are generalized to arbitrary dimension and codimension in [9].

Remark 8.2. The case when $*\omega_1 > 0$ and $*\omega_2 = 0$ corresponds to Σ is Lagrangian surface with respect to the symplectic form ω_2. If Σ is the graph of a map f, then f is indeed an area preserving diffeomorphism. This case and the application to the structure of the diffeomorphism groups of compact Riemann surfaces are discussed in [8].

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